Fuzzy logic as a logic of the expressive strength of information

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Abstract

We develop alternative semantics for Łukasiewicz logic and for cancellative hoop logic according to the following idea. We formalize statements reflecting an inexact knowledge of certain (sharp) properties; we assume that all what can be known about a property is its expressive strength.

To this end, we consider a Boolean algebra endowed with an automorphism group or, alternatively, with a measure. The Boolean algebra is meant to model a collection of properties; and the additional structure is used to identify pairs of properties which, although possibly distinct, are equally strong. Propositions are defined as subsets of the algebra containing with any element also those identified with it in this way. We show that then, the set of all propositions carries the structure of an MV-algebra or of a cancellative hoop.

1 Introduction

This note is meant as a contribution to the discussion about the interpretation of fuzzy logics. Recall that propositions in fuzzy logics are usually interpreted by values from the real unit interval. Whereas it is relatively easy to explain what the extreme values 0 (if used) and 1 stand for and why we use the reals' natural order, the meaning of specific elements strictly between 0 and 1 is as difficult to explain as to justify particular choices of the connectives. There have been several attempts to clarify the picture; for a compilation of different approaches we refer to J. Paris's article [Par].

The fuzzy logics discussed here are among those which are based on continuous t-norm algebras; namely, as the set of truth values, we take the closed or half-open real unit interval, which is endowed with a continuous t-norm, the corresponding residuum, the constant 1 and, if included in the set of truth values, the constant 0. More specifically, we deal with the Łukasiewicz logic **ŁL** [COM, Haj] and the cancellative hoop logic **CHL** [EGHM], whose corresponding varieties consist of MV-algebras and cancellative hoops, respectively.

Our approach to the interpretational issue is to search for natural representations of the algebras contained in these varieties. We wonder if MV-algebras or cancellative hoops naturally arise in some simple framework. The aim is to base fuzzy logics on structures whose meaning is more evident than the meaning of the structures underlying the canonical semantics.

In the present paper, we propose a specific way to model uncertainty of information. Let a Boolean algebra represent a system of sharp properties arising in a given context. We consider the case that a property might not be specifiable uniquely; we rather assume that not more than its expressive strength can be known. To this end, we either endow the Boolean algebra with a fixed automorphism group; a proposition is then a subset of the algebra closed under the action of this group. Or we simply endow the Boolean algebra with a measure; a proposition is in this case a subset containing with any element also those of the same measure.

On the collection of all proposition defined in the former or latter way, we define two binary operations. Namely, two proposition may be connected by pointwise infimum; in the given context, this operation is probably the most natural choice for a conjunction. A second operation which we define reflects the nature of an implication. Depending on the details, we are led to the structure of an MV-algebra or a cancellative hoop; we actually even get an isomorphic copy of the standard t-norm algebras. So all in all, we offer alternative semantics for **ŁL** or **CHL** based on the notion of uncertainty.

2 Preliminaries

We consider in this paper two kinds of fuzzy logics: the Łukasiewicz propositional logic **ŁL** [COM, Haj] and the cancellative hoop propositional logic **CHL** [EGHM]. We collect the basic facts about them.

The language of **ŁL** is $\{\odot, \rightarrow, 0, 1\}$; an **ŁL**-formula is built up from atomic formulas $\varphi_0, \varphi_1, \ldots$ and the constants 0 and 1 by means of the binary connectives $\odot$ and $\rightarrow$. The language of **CHL** is $\{\odot, \rightarrow, 1\}$; a **CHL**-formula is an **ŁL**-formula in which the constant 0 does not appear. (Note that usually the constant 1 is not included in the language; we do so here for reasons of clarity.)

An evaluation for **ŁL** in a model $(M; \underline{\odot}, \underline{\rightarrow}, \underline{0}, \underline{1})$ is a mapping from the **ŁL**-formulas to M such that $v(\alpha \odot \beta) = v(\alpha) \underline{\odot} v(\beta)$ and $v(\alpha \rightarrow \beta) = v(\alpha) \underline{\rightarrow} v(\beta)$ for all $\alpha, \beta \in \mathcal{F}$, and furthermore $v(0) = \underline{0}$ and $v(1) = \underline{1}$. An **ŁL**-formula is called valid in M if it is assigned $\underline{1}$ by all evaluations in M. Similarly, we define evaluations and validity of **CHL**-formulas in models $(M; \underline{\odot}, \underline{\rightarrow}, \underline{1})$.

The Łukasiewicz algebra is $([0,1]; \odot_L, \rightarrow_L, 0, 1)$, where $[0,1]$ is the real unit interval and $\odot_L$ and $\rightarrow_L$ are the Łukasiewicz conjunction and implication, respectively:

$$\begin{aligned} \odot_L &\colon\ [0,1]^2 \rightarrow [0,1], \quad (a,b) \mapsto (a+b-1) \vee 0, \\ \rightarrow_L &\colon\ [0,1]^2 \rightarrow [0,1], \quad (a,b) \mapsto (1-a+b) \wedge 1. \end{aligned} \tag{1}$$

Similarly, the standard cancellative hoop is $((0,1]; \odot_P, \rightarrow_P, 1)$, where $\odot_P$ and $\rightarrow_P$ are the product conjunction and implication, respectively:

$$\begin{aligned} \odot_P &\colon\ (0,1]^2 \rightarrow (0,1], \quad (a,b) \mapsto a \cdot b, \\ \rightarrow_P &\colon\ (0,1]^2 \rightarrow (0,1], \quad (a,b) \mapsto \begin{cases} \frac{b}{a} & \text{if } a \geq b, \\ 1 & \text{else.} \end{cases} \end{aligned} \tag{2}$$

A formula α is called valid in **ŁL** or **CHL** if α is valid in the Łukasiewicz algebra or the standard cancellative hoop, respectively.

For an axiomatization of these logics as well as for any further details, we refer to [COM] and [EGHM].

We will be furthermore concerned mostly with the theory of Boolean algebras and their automorphism groups. For basic facts on this subject, we refer to

[Sik] or [Mon]. For automorphisms of Boolean algebras, see [Fre, Ch. 38]. For group invariant measures on Boolean algebras, see [Fre, Ch. 39].

A Boolean algebra will always assumed to be non-trivial, that is, to contain at least two elements.

We will denote the complementation in a Boolean algebra by $^\perp$, and we will express disjointness, i.e. zero-infimum, of a pair a and b by $a \perp b$. As usual, we will call a Boolean algebra separable if it has a countable dense subset.

Finally, recall that a Boolean algebra $\mathcal{B}$ may be isomorphically and densely embedded into a complete Boolean algebra, in a way that all infima and suprema existing in $\mathcal{B}$ are preserved.

3 Boolean algebras with an automorphism group

We shall develop a model for reasoning with uncertain information of a special type. Let us explain the underlying idea. Assume that we are given a countable collection of yes-no properties logically related in any specific way. An appropriate model will be a separable Boolean algebra $\mathcal{B}$. Let us furthermore assume that we may observe the properties from different perspectives, but that we have to communicate them without reference to a specific viewpoint. We may find it then appropriate to endow $\mathcal{B}$ with a "symmetry group" G, whose intended meaning is that we cannot distinguish between properties modeled by an element $a \in \mathcal{B}$ and its image under an automorphism from G. Statements may then be modelled by subsets of $\mathcal{B}$ closed under the action of G.

In this situation, we suppose to be still able to classify the properties according to their strength; recall that a property is stronger if modelled by a smaller element. Consequently, we will not assume to be given an arbitrary automorphism group, but a group which, most important, does never map an element a to an element strictly below a.

Definition 3.1 Let $(\mathcal{B}; \wedge, \vee, ^\perp, 0, 1)$ be a separable Boolean algebra, and let G be a group of automorphisms of $\mathcal{B}$. For $a, b \in \mathcal{B}$, let $a \sim b$ if $g(a) = b$ for some $g \in G$; and let $a \preceq b$ if there is an $a' \sim a$ such that $a' \leq b$.

We will say that G acts on $\mathcal{B}$ *homogeneously* if the following conditions hold:

(G1) For any $a_1, a_2, b_1, b_2 \in \mathcal{B}$ such that $a_1 \perp a_2$ and $b_1 \perp b_2$ as well as $a_1 \sim b_1$ and $a_2 \sim b_2$, we have $a_1 \vee a_2 \sim b_1 \vee b_2$.

(G2) Let $b_1, b_2, \ldots \in \mathcal{B}$ be such that $\bigwedge_i b_i = 0$. Then from $a \preceq b_i$ for every i, it follows $a = 0$.

(G3) For any $a, b \in \mathcal{B}$, there is a $g \in G$ such that a and $g(b)$ are comparable.

The meaning of condition (G1) is obvious: if two elements are piecewise related by $\sim$, then the elements are related by $\sim$ themselves. From (G1) it follows in particular that $g(a) \leq a$ implies $g(a) = a$, where $a \in \mathcal{B}$ and $g \in G$. Indeed, $g(a) < a$ would mean $1 = a \vee a^{\perp} \sim g(a) \vee a^{\perp} < 1$, which is impossible. The second condition, (G2), can be viewed as continuity at 0. Condition (G3), finally, is an analogue of the notion of transitivity. Note that (G3) is equivalent to saying that $a \preceq b$ or $b \preceq a$ for any $a, b \in \mathcal{B}$.

We will see that the conditions (G1)–(G3) have a convenient consequence: they ensure the existence of a G-invariant measure. By a measure m on a Boolean algebra, we mean a mapping $m\colon \mathcal{B} \to [0,1]$ such that $m(a \vee b) = m(a) + m(b)$ for disjoint elements $a, b \in \mathcal{B}$, and $m(1) = 1$. m is called strictly positive if $m(a) > 0$ for all $a > 0$. Furthermore, m is called G-invariant if $m(g(a)) = m(a)$ for all $a \in \mathcal{B}$ and $g \in G$.

The following theorem is a special version of Y. Kawada's fundamental theorem on automorphism groups on Boolean algebras; see [Kaw] or [Fre, Ch. 394]. Recall that a group G acting on a Boolean algebra $\mathcal{B}$ is called ergodic if for any non-zero $a, b \in \mathcal{B}$, $a \wedge g(b) > 0$ for some $g \in G$. Furthermore note that a sequence $(a_i)_{i<\omega}$ is called a partition of unity if the a_i are pairwise disjoint and $\bigvee_i a_i = 1$.

Theorem 3.2 *Let $\mathcal{B}$ be a complete separable Boolean algebra, and let G be an ergodic group of automorphisms. Assume that, for any partition of unity $(a_i)_{i<\omega}$ and pairwise disjoint elements b_i, $i < \omega$, such that $a_i \sim b_i$ for every i, $(b_i)_{i<\omega}$ is a partition of unity as well. Then there exists a unique G-invariant, strictly positive measure m on $\mathcal{B}$.*

We will apply this theorem to our setting.

Lemma 3.3 *Let $\mathcal{B}$ be a separable Boolean algebra, and let G be an automorphism group of $\mathcal{B}$ acting homogeneously on $\mathcal{B}$. Let $(a_i)_{i<\omega}$ be a partition of unity, and let b_i, $i < \omega$, be pairwise disjoint elements such that $a_i \sim b_i$ for every i. Then $(b_i)_{i<\omega}$ is a partition of unity as well.*

Proof. Let $c_i = (a_1 \vee \ldots \vee a_i)^\perp$ and $d_i = (b_1 \vee \ldots \vee b_i)^\perp$, $i < \omega$. Then, for every i, we conclude from (G1) that $c_i \sim d_i$. Assume that $e \geq b_i$ for all i; then $e^\perp \leq d_i$ for every i. Because $\bigwedge_j c_j = 0$, it follows from (G2) that $e^\perp = 0$, whence $e = 1$. So $\bigvee_j b_j = 1$. □

Theorem 3.4 *Let $\mathcal{B}$ be a separable Boolean algebra, and let G be an automorphism group of $\mathcal{B}$ acting homogeneously on $\mathcal{B}$. Then there exists a unique G-invariant, strictly positive measure m on $\mathcal{B}$.*

Proof. Let $\overline{\mathcal{B}}$ be the completion of $\mathcal{B}$, and consider $\mathcal{B}$ as a subalgebra of $\overline{\mathcal{B}}$. Since $\mathcal{B}$ is dense in $\overline{\mathcal{B}}$, the latter algebra is still separable. Moreover, every $g \in G$ is uniquely extendable from $\mathcal{B}$ to an automorphism $\overline{g}$ of $\overline{\mathcal{B}}$; we denote by $\overline{G}$ the group of all $\overline{g}$ for $g \in G$.

For non-zero $a, b \in \overline{\mathcal{B}}$, there are non-zero $a' \leq a$ and $b' \leq b$ in $\mathcal{B}$. By (G3), $a' \leq g(b')$ or $g(b') \leq a'$ for some $g \in G$; then $a \wedge g(b) \geq a' \wedge g(b') > 0$, that is, $\overline{G}$ is ergodic.

Let $(a_i)_{i<\omega}$ be a partition of unity in $\overline{\mathcal{B}}$, and let $g_i \in G$, $i < \omega$, be such that $\overline{g}_i(a_i)$, $i < \omega$ are pairwise disjoint. For every i, let $a_{ij} \in \mathcal{B}$, $j < \lambda_i \leq \omega$, be pairwise disjoint and such that $a_i = \bigvee_{j<\lambda_i} a_{ij}$. Then $(a_{ij})_{i<\omega, j<\lambda_i}$ is a partition of unity in $\overline{\mathcal{B}}$ and consequently also in $\mathcal{B}$. Furthermore, $\overline{g}_i(a_{ij}) = g_i(a_{ij})$, $i < \omega$, $j < \lambda_i$ are pairwise disjoint in $\overline{\mathcal{B}}$ as well as in $\mathcal{B}$. It follows from Lemma 3.3 that $\bigvee_{i,j} g_i(a_{ij}) = 1$ in $\mathcal{B}$, so also in $\overline{\mathcal{B}}$; thus $\bigvee_i g_i(a_i) = 1$ in $\overline{\mathcal{B}}$.

So by Theorem 3.2, there is a unique G-invariant, strictly positive measure m on $\overline{\mathcal{B}}$, and the assertion follows. □

This theorem implies the following important fact.

Lemma 3.5 *Let $\mathcal{B}$ be a separable Boolean algebra, and let G be an automorphism group of $\mathcal{B}$ acting homogeneously on $\mathcal{B}$. Let $m : \mathcal{B} \to [0, 1]$ be* *the G-invariant, strictly positive measure on $\mathcal{B}$. Then $a \sim b$ if and only if $m(a) = m(b)$; and $a \preceq b$ if and only if $m(a) \leq m(b)$.*

In particular, the equivalence relation $\sim$ is compatible with $\preceq$, and the relation induced by $\preceq$ on the set of $\sim$-equivalence classes, is a bounded total order.

Moreover, if $a \leq c$ and $a \preceq b \preceq c$, then $a \leq b' \leq c$ for some $b' \sim b$.

Proof. Let $a, b \in \mathcal{B}$. If $a \sim b$, then $m(a) = m(b)$ by the G-invariance. Conversely, if $m(a) = m(b)$, there is an $g \in G$ such that $g(a) \leq b$ or $g(a) \geq b$. By strict positivity, $g(a) = b$, that is, $a \sim b$. So the first claim is clear.

Taking into account that $a \preceq b$ or $b \preceq a$ for any $a, b \in \mathcal{B}$, it moreover follows that $a \preceq b$ if and only if $m(a) \leq m(b)$. So the first part of the lemma is proved.

It further follows that $a \preceq b$ and $b \preceq a$ implies $a \sim b$. So $\preceq$ induces a partial order on the quotient of $\mathcal{B}$ w.r.t. $\sim$. Clearly, this order is bounded, and by (G3), it is total.

Let finally $a \leq c$ and $a \preceq b \preceq c$. Then $a \leq b''$ for some $b'' \sim b$, and $b'' \wedge a^\perp \preceq c \wedge a^\perp$. So $d \leq c \wedge a^\perp$ for some $d \sim b'' \wedge a^\perp$, and putting $b' = d \vee a$, we get $b \sim (b'' \wedge a^\perp) \vee a \sim b'$ and $a \leq b' \leq c$. $\square$

We shall now formalize the concept of a proposition in the context of a Boolean algebra endowed with an automorphism group. According to the ideas explained at the beginning of the section, a proposition should be a subset of the Boolean algebra closed under the action of the group. In addition to that, we will require that this subset contains also all elements representing weaker properties than those already present; that is, we will require it to be an order-filter. A last condition will ensure that a proposition does not contain the zero element, which represents contradiction.

Asserting a proposition $\varphi \subseteq \mathcal{B}$ should consequently be understood as follows: *For any $a \in \varphi$, not necessarily a itself holds, but $g(a)$ for some $g \in G$.*

Definition 3.6 Let $\mathcal{B}$ be a separable Boolean algebra, and let G be an automorphism group of $\mathcal{B}$ acting homogeneously on $\mathcal{B}$. We call a non-empty set $\varphi \subseteq \mathcal{B}$ a $(\mathcal{B}, G)$*-proposition* if for $a, b \in \mathcal{B}$

(i) from $a \in \varphi$ and $b \geq a$ it follows $b \in \varphi$,

(ii) from $a \in \varphi$ it follows $g(a) \in \varphi$ for any $g \in G$,

(iii) $0 \notin \varphi$.

Let $\mathcal{F}$ be the set of all $(\mathcal{B}, G)$-propositions. For $\varphi, \psi \in \mathcal{F}$, let

$$\begin{aligned} \varphi \odot \psi &= \{a \wedge b \colon a \in \varphi,\ b \in \psi,\ a \wedge b > 0\}; \\ \varphi \to \psi &= \{c \colon \text{for every } a \in \varphi,\ \ a \wedge c \in \varphi \cup \psi\}, \end{aligned} \tag{3}$$

and set

$$\begin{aligned} \mathbf{0} &= \mathcal{B} \setminus \{0\}, \\ \mathbf{1} &= \{1\}. \end{aligned} \tag{4}$$

Then $(\mathcal{F}; \odot, \to, \mathbf{0}, \mathbf{1})$ is called the *proposition algebra* associated to $(\mathcal{B}, G)$.

Note that $\odot$ and $\to$ are indeed operations on $\mathcal{F}$ and that $\mathbf{0}, \mathbf{1} \in \mathcal{F}$.

When the reference to a pair $(\mathcal{B}, G)$ is clear, we will speak simply about propositions. We remark that $\mathbf{0}$ does not play, as usual, the role of the false or contradictory proposition, but rather the role of the strongest non-contradictory one.

Lemma 3.7 *Let $\mathcal{B}$ be a separable Boolean algebra, and let G be an automorphism group of $\mathcal{B}$ acting homogeneously on $\mathcal{B}$. Let m be the G-invariant, strictly positive measure on $\mathcal{B}$; let $M = \operatorname{ran} m$, i.e. the range of m, and let $\overline{M}$ be its closure in $[0,1]$. Then the propositions are exactly the subsets of the form $\{a \in \mathcal{B} \colon m(a) > r\}$ for $r \in \overline{M} \setminus \{1\}$ or $\{a \in \mathcal{B} \colon m(a) \geq r\}$ for $r \in M \setminus \{0\}$.*

Proof. By Lemma 3.5, we know that for any $a, b \in \mathcal{B}$, $a \sim b$ if and only if $m(a) = m(b)$. It follows that any set of the indicated form is indeed a proposition.

Conversely, let φ be a proposition, and put $r = \inf \{m(a) \colon a \in \varphi\}$. Then either $r = m(a)$ for some $a \in \mathcal{B}$; in this case, $\varphi = \{a \colon m(a) \geq r\}$. Or r is strictly smaller than all $m(a)$; then, $\varphi = \{a \colon m(a) > r\}$. □

We will next describe the structure of the set of $(\mathcal{B}, G)$-propositions for some pair $\mathcal{B}$ and G, that is, the algebra $(\mathcal{F}; \odot, \to, \mathbf{0}, \mathbf{1})$. We will treat the finite and the infinite case separately.

Let us denote by L_k, $k = 0, 1, \ldots$, the $k+1$-element MV-algebra. So $L_k = \{0, \frac{1}{k}, \ldots, 1\}$ for $k \geq 1$, endowed with the operations $\odot_L$ and $\rightarrow_L$ given by the same formulas as in (1).

Theorem 3.8 *Let $\mathcal{B}$ be a finite Boolean algebra, and let G be an automorphism group of $\mathcal{B}$ acting homogeneously on $\mathcal{B}$. Then $(\mathcal{F}; \odot, \rightarrow, \mathbf{0}, \mathbf{1})$ is isomorphic to the MV-algebra L_{k-1}, where k is the number of atoms of $\mathcal{B}$.*

Proof. Since automorphisms of finite Boolean algebras are simply permutations of the atoms, it follows from (G3) and (G1) that, for $a, b \in \mathcal{B}$, $a \sim b$ if and only if the number of atoms below a and b coincide. It furthermore follows that m counts the number of atoms; ran $m = \{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$. So the propositions are the subsets $\varphi_i = \{a \colon m(a) \geq \frac{i+1}{k}\}$ for $i = 0, \ldots, k-1$, that is, φ_i contains all those elements of $\mathcal{B}$ which cover at least $i+1$ atoms.

Clearly, $\varphi_0 = \mathbf{0} = \{a > 0\}$ and $\varphi_{k-1} = \mathbf{1} = \{1\}$. Furthermore, we calculate $\varphi_i \odot_L \varphi_j = \varphi_{(i+j-(k-1)) \vee 0}$ and $\varphi_i \rightarrow_L \varphi_j = \varphi_{((k-1)-i+j) \wedge (k-1)}$; we leave out the details. This proves the assertion. $\square$

In the infinite case, we get a structure embeddable into the Alexandrov double-arrow space. Let Q be a dense subset of the real unit interval such that Q contains 0 and 1; then define

$$\begin{aligned} \tilde{Q} = \{(r,d) \in [0,1] \times \{0,1\} \colon \quad & \text{if } r = 0 \text{ or } r \in [0,1] \setminus Q, \text{ then } d = 1; \\ & \text{if } r = 1, \text{ then } d = 0\}; \end{aligned}$$

$[0,1]$ and $\{0,1\}$ being given the natural order, we endow $\tilde{Q}$ with the lexicographical order.

Theorem 3.9 *Let $\mathcal{B}$ be an infinite separable Boolean algebra, and let G be an automorphism group of $\mathcal{B}$ acting homogeneously on $\mathcal{B}$. Let $Q =$ ran m, where m is the G-invariant, be a strictly positive measure on $\mathcal{B}$. Then Q is closed under the operations $\odot_L$ and $\rightarrow_L$, and $0, 1 \in Q$. For $(r,d), (s,e) \in \tilde{Q}$,* *let*

$$(r,d)\odot(s,e)=\begin{cases}(r\odot_L s, d\vee e) & \textit{if } r+s>1,\\ (0,1) & \textit{if } r+s\leq 1;\end{cases}$$

$$(r,d)\to(s,e)=\begin{cases}(r\to_L s,(1-d)\wedge e) & \textit{if } r>s \textit{ and } r\to_L s\in Q\setminus\{0\},\\ (r\to_L s,1) & \textit{if } r>s \textit{ and } r\to_L s\notin Q\setminus\{0\},\\ (1,0) & \textit{if } r\leq s.\end{cases} \tag{5}$$

Then $(\mathcal{F};\odot,\to,\mathbf{0},\mathbf{1})$ *is isomorphic to* $(\tilde{Q};\odot,\to,(0,1),(1,0))$.

Proof. Since $\mathcal{B}$ is not finite, Q is a dense subset of the real unit interval containing 0 and 1. Furthermore, Q is closed under complementation and under addition in case the sum is ≤ 1; so Q is closed under $\odot_L$ and $\to_L$. It is then not difficult to see that $\odot$ and $\to$ are well defined on $\tilde{Q}$.

For $(r,d)\in\tilde{Q}$, let

$$\varphi_{(r,d)}=\begin{cases}\{a\in\mathcal{B}\colon\ m(a)\geq r\} & \text{if } d=0,\\ \{a\in\mathcal{B}\colon\ m(a)>r\} & \text{if } d=1;\end{cases}$$

by Lemma 3.7, $\mathcal{F}=\{\varphi_{(r,d)}\colon\ (r,d)\in\tilde{Q}\}$ then.

We have $\varphi_{(0,1)}=\mathbf{0}$ and $\varphi_{(1,0)}=\mathbf{1}$. We need to check that the mapping $\tilde{Q}\to\mathcal{F},\ \ (r,d)\mapsto\varphi_{(r,d)}$ preserves $\odot$ and $\to$, defined by (5) and (3), respectively. So let $(r,d),(s,e)\in\tilde{Q}$, and let us calculate $\varphi_{(r,d)}\odot\varphi_{(s,e)}$ and $\varphi_{(r,d)}\to\varphi_{(s,e)}$ according to (3). We restrict to the case $d=1$ and $e=0$; the other cases work similarly.

We have $s>0$. If $r=0$, then clearly $\varphi_{(r,1)}\odot\varphi_{(s,0)}=\varphi_{(0,1)}$. Assume $r>0$. If $r+s\leq 1$, there is, for any $t\in\operatorname{ran} m$ such that $0<t\leq r\wedge s$, a pair $a,b\in\mathcal{B}$ such that $m(a)>r$, $m(b)\geq s$, and $m(a\wedge b)=t$. It follows $\varphi_{(r,1)}\odot\varphi_{(s,0)}=\mathcal{B}\setminus\{0\}=\varphi_{(0,1)}$.

If $r+s>1$, $m(a\wedge b)>r+s-1$ for any a and b such that $m(a)>r$ and $m(b)\geq s$. Furthermore, for any c such that $m(c)>r+s-1$ there are elements $a,b\in\mathcal{B}$ such that $m(a)>r$, $m(b)\geq s$, and $a\wedge b=c$. It follows $\varphi_{(r,1)}\odot\varphi_{(s,0)}=\{a\in\mathcal{B}\colon\ m(a)>r+s-1\}=\varphi_{(r+s-1,1)}$.

We now turn to the implication. Note that $\varphi_{(r,1)}\cup\varphi_{(s,0)}=\varphi_{(r,1)\wedge(s,0)}$. If $r<s$, we have $\varphi_{(r,1)}\to\varphi_{(s,0)}=\{c\colon$ for any a, $m(a)>r$ implies $m(a\wedge c)>r\}$;

we readily see that this set contains only $c = 1$, that is, $\varphi_{(r,1)} \to \varphi_{(s,0)} = \varphi_{(1,0)}$ then. If $s \leq r$, we have $\varphi_{(r,1)} \to \varphi_{(s,0)} = \{c :$ for any a, $m(a) > r$ implies $m(a \wedge c) \geq s\}$. If then $m(c) \geq 1 - r + s$, we have $m(a \wedge c) \geq s$ for any a such that $m(a) > r$, and if $m(c) < 1 - r + s$, there is an a such that $m(a) > r$ and $m(a \wedge c) < s$. So $\varphi_{(r,1)} \to \varphi_{(s,0)} = \varphi_{(1-r+s,0)}$, provided that $1 - r + s \in \text{ran } m \setminus \{0\} = Q \setminus \{0\}$; else $\varphi_{(r,1)} \to \varphi_{(s,0)} = \varphi_{(1-r+s,1)}$. $\square$

As a straightforward consequence of Theorems 3.8 and 3.9, we arrive at the main theorem.

Theorem 3.10 *Let α be an* **ŁL***-formula. Then the following statements are equivalent:*

(i) *For any separable Boolean algebra $\mathcal{B}$ and automorphism group G acting homogeneously on $\mathcal{B}$, α is valid in the associated proposition algebra $(\mathcal{F}, \odot, \to, \mathbf{0}, \mathbf{1})$.*

(ii) *α is valid in* **ŁL**.

Proof. Let α be valid in **ŁL**, and let $\mathcal{B}$ and G be as specified in (i). If $\mathcal{B}$ is finite, α is valid in $\mathcal{F}$ by Theorem 3.8. If $\mathcal{B}$ is infinite, then any evaluation of the $\mathcal{L}$-formulas in $\tilde{Q}$, where $\tilde{Q}$ is as specified in Theorem 3.9, assigns to α the element $(1, 0)$, because according to Theorem 3.9 the connectives are interpreted in the Łukasiewicz algebra w.r.t. first component, and $(1, 0)$ is the only element whose first component is 1.

Let α be not valid in **ŁL**, and let v be any evaluation of the $\mathcal{L}$-formulas in the Łukasiewicz algebra such that $v(\alpha) < 1$. We may use either Theorem 3.8 or 3.9 to determine a pair $\mathcal{B}$ and G as specified in (i), and an evaluation in the associated proposition algebra which does not assign α the top element. $\square$

Note that we showed more than asserted; we may restrict the statement (i) in the way that the Boolean algebra is finite or, alternatively, that it is infinite.

4 Boolean algebras with a measure

We will in this section present an alternative way to formalize reasoning with uncertain information. According to our concept, we consider properties just with respect to their strength. Rather than relating properties of equal strength by an automorphism group, we may express strength in a more direct way: by a measure. This is what we will do now; we will work with the pair of a Boolean algebra and a measure on it.

Whereas this approach might be less appealing than the preceding one, the advantage is that we may easily cover alternatively **ŁL**, the Łukasiewicz logic, or **CHL**, the cancellative hoop logic.

Note that **CHL** is similar to the better-known product logic. However, the latter is based on a t-norm algebra which is not ordinally irreducible; namely, it is the ordinal sum of the standard cancellative hoop and the two-element residuated lattice. For this reason, finding representations of this algebra is more difficult. In the present case, we would for instance have to restrict to appropriate subuniverses of the Boolean algebra; we might discuss this possibility in a subsequent paper.

In this section, we drop the restriction that the range of a measure is within $[0, 1]$; a measure on a Boolean algebra $\mathcal{B}$ is rather assumed to be a function $m: \mathcal{B} \to \mathbb{R}^+ \cup \{\infty\}$ which is additive for disjoint elements. In case that $m(1) < \infty$, we say that m is finite; otherwise m is called infinite.

For elements a, b of a Boolean algebra, we say that a *touches* b, denoted by $a \bowtie b$, if $a \not\leq b^\perp$.

Definition 4.1 Let $\mathcal{B}$ be a separable Boolean algebra, and let m be a strictly positive measure on $\mathcal{B}$. We shall call m *homogeneous* if, for all $a, b, c \in \mathcal{B}$, from $a \leq c$ and $m(a) \leq m(b) \leq m(c)$, it follows that $m(b) = m(b')$ for some b' such that $a \leq b' \leq c$.

A set of the form

$$\varphi = \{a \in \mathcal{B} \colon\ m(a^\perp) \leq r\ \}$$

for some $r \in \operatorname{ran} m \cap \mathbb{R}^+$, is called a $(\mathcal{B}, m)$-*proposition*.

Let $\mathcal{F}$ be the set of all $(\mathcal{B}, m)$-propositions. For $\varphi, \psi \in \mathcal{F}$, define

$$\begin{aligned} \varphi \odot \psi &= \{a \wedge b \colon\ a \in \varphi,\ b \in \psi\}; \\ \varphi \to \psi &= \{c \in \mathcal{B} \colon \text{ for every } a \in \varphi,\ \ a \bowtie c \ \text{ and } \ a \wedge c \in \varphi \cup \psi\}, \end{aligned}$$

and set

$$\begin{aligned} \mathbf{0} &= \mathcal{B}, \\ \mathbf{1} &= \{1\}. \end{aligned}$$

The *proposition algebra* associated to $(\mathcal{B}, m)$ is $(\mathcal{F}; \odot, \to, \mathbf{0}, \mathbf{1})$ if m is finite, and otherwise $(\mathcal{F}; \odot, \to, \mathbf{1})$.

Note that propositions consist of elements of the Boolean algebra whose complement is bounded in measure, rather than the elements themselves. The idea is that when measuring the expressiveness of a property, we should measure what is excluded – not what remains possible. However, if the measure is finite, both possibilities amount to the same.

We give the main statements of this section straightforwardly. The argumentation is easier than in the preceding section, and we skip the proofs.

Theorem 4.2 *Let α be an* **ŁL***-formula. Then the following statements are equivalent:*

(i) *For any separable Boolean algebra $\mathcal{B}$ and finite homogeneous, strictly positive measure m on $\mathcal{B}$, α is valid in the associated proposition algebra $(\mathcal{F}; \odot, \to, \mathbf{0}, \mathbf{1})$.*

(ii) *α is valid in* **ŁL***.*

Similarly, we have:

Theorem 4.3 *Let α be a* **CHL***-formula. Then the following statements are equivalent:*

(i) *For any separable Boolean algebra $\mathcal{B}$ and infinite homogeneous, strictly positive measure m on $\mathcal{B}$, α is valid in the associated proposition algebra $(\mathcal{F}; \odot, \to, \mathbf{1})$.*

(ii) *α is valid in* **CHL***.*

5 An illustrative example

We give in this section an easy example. It refers to the formalism explained in Section 3, based on Boolean algebras with an automorphism group; but it may be regarded mutatis mutandis as an example of the formalism described in the preceding Section 4 as well.

Assume that we are given a countable collection of yes-no properties which are logically unrelated. The appropriate model is apparently the Boolean algebra $\mathcal{B}$ freely generated by λ elements, say by $\{e_i : i < \lambda\}$, where $\lambda \leq \omega$ is the number of properties to be modelled.

Call an automorphism g of $\mathcal{B}$ *of finite support* if $g(e_i) = e_i$ for all but finitely many $i < \lambda$. Let G be the group of all automorphisms of $\mathcal{B}$ of finite support.

Let us consider the pair $(\mathcal{B}, G)$. There is a natural measure on $\mathcal{B}$, given by $m(e_{i_1}^{\pm} \wedge \ldots \wedge e_{i_k}^{\pm}) = (\frac{1}{2})^k$, where $i_1, \ldots, i_k$ are pairwise distinct indices and $^{\pm}$ is individually chosen as $^{\perp}$ or the identity. It is not difficult to see that $a \in \mathcal{B}$ is mapped to a' by some $g \in G$ if and only if $m(a) = m(a')$. In particular, G acts homogeneously on $\mathcal{B}$. The propositions are all sets $\{a : m(a) \geq r\}$ for $r = n(\frac{1}{2})^k$, where $k < \omega$ and $1 \leq n \leq 2^k$, and all sets $\{a : m(a) > r\}$ for all $r \in [0, 1)$.

Endow now $\mathcal{F}$, the set of propositions, with the operations $\odot$ and $\rightarrow$ as defined in Definition 3.6, and with the constants $\mathbf{0} = \mathcal{B} \setminus \{0\}$ and $\mathbf{1} = \{1\}$. By identifying two propositions in the case that the infimum of the measure of their elements coincides, we get the Łukasiewicz algebra.

At this point, we may notice conceptual similarities to certain other approaches to the problem how to interpret statements of fuzzy logics. Namely, we may consider the fuzzy propositions just defined as sets of formulas of Boolean propositional logic, $e_1, \ldots$ being the atomic formulas. Furthermore, if λ is finite, we may associate to every fuzzy proposition φ the proportion of those two-valued truth assignments under which a given $a \in \varphi$ is true, among all truth assignments. Note that this proportion is the same for all $a \in \varphi$ and that it is characteristic for φ.

A similar mapping appears in connection with C. Fermüller's investigations about the relationship between the three standard fuzzy logics on the one hand and an approach to vagueness called supervaluationism on the other hand [Fer]. Namely, in [Fer], it is proposed to interpret an atomic formula e

by the proportion of (certain) two-valued truth assignments under which e is true. Compound formulas, however, are interpreted in a different manner, namely on the base of game semantics, which is appropriately adapted for each of the discussed fuzzy logics. Indeed, otherwise a problem would occur which is characteristic also for voting semantics, cf. e.g. [Par]; evaluations based on measuring the proportion of some fixed set are not truth-functional. We may mention, incidentally, that it was actually these latter difficulties which led to the present work.

6 Conclusion

In our paper, an abstract framework is introduced which provides a connection between the notion of uncertainty on the one hand and formal fuzzy calculi on the other hand. In this framework, entities appear which may be considered as propositions in fuzzy logic and which connect exactly in the way proposed by one out of two well-known fuzzy logics – namely, Łukasiewicz logic or cancellative hoop logic.

Namely, we endow a Boolean algebra, modelling a set of sharp properties, either with an automorphism group or with a measure, in both cases with the intention to identify any two properties which are possibly not comparable, but have the same weight when asserted. Propositions are then order-ideals closed under the relation of being identifiable. Under appropriate assumptions, the set of all propositions carries in a natural way the structure of the respective standard algebra.

This work is meant as a first step towards establishing alternative semantics for fuzzy logics which is no longer based on the real unit interval. Its continuation is projected into several directions. Certainly, it would be desirable to cover more kinds of fuzzy logics – like Hájek's Basic Logic. Furthermore, the aim is to provide a formalism based on as simple notions as possible. In particular, conditions like (G2), which are infinitary, should be avoided if possible.

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