Unit 4

The Extension Principle

Images and Preimages of Functions

Let $f : X \to Y$ be a function and A be a subset of X. Then the *image of* A *w.r.t.* f is defined as follows:

$$f(A) = \{y \in Y \mid \text{ there is an } x \in A \text{ such that } y = f(x)\}$$

Let B be a subset of Y. Then the *preimage of* Y *w.r.t.* f is defined as

$$f^{-1}(B) = \{x \in X \mid \text{ there is a } y \in B \text{ such that } y = f(x)\}.$$

Question: *How can we generalize this to fuzzy sets?*

The Extension Principle

For a given function $f : X \to Y$, we define a fuzzy relation R as

$$\mu_R(x, y) = \begin{cases} 1 & \text{if } y = f(x) \\ 0 & \text{otherwise} \end{cases}$$

Let A be a fuzzy set on X and B be a fuzzy set on Y. Then we can compute $\hat{f}(A)$ and $\hat{f}^{-1}(B)$ as $R_T(A)$ and $R_T^{-1}(B)$, respectively, which simplify to

$$\mu_{\hat{f}(A)}(y) = \sup\{\mu_A(x) \mid y = f(x)\},$$
$$\mu_{\hat{f}^{-1}(B)}(x) = \sup\{\mu_B(y) \mid y = f(x)\}.$$

Representation by Means of α-Cuts

For $\alpha \in [0, 1[$, the *strict* α*-cut* of a fuzzy set A on X is a crisp set defined as

$$A^{>\alpha} = \{x \in X \mid \mu_A(x) > \alpha\}.$$

For a given function $f : X \to Y$, the following holds for all $\alpha \in [0, 1[$:

$$\hat{f}(A)^{>\alpha} = f(A^{>\alpha})$$

Example

$$X = \{a, b, c, d, e\},\ Y = \{r, s, t, u\}$$

x	$\mu_A(x)$
a	0.6
b	0.4
c	0.1
d	0.0
e	0.3

x	$f(x)$
a	r
b	s
c	r
d	t
e	s

y	$\mu_B(y)$
r	0.0
s	0.3
t	0.7
u	0.1

$\hat{f}(A) = ?$ $\hat{f}^{-1}(B) = ?$

Extension Principle for Cartesian Products

Suppose we are given a function $f : X_1 \times \cdots \times X_n \longrightarrow Y$, and fuzzy sets A_i on the respective X_i (for $i = 1, \ldots, n$). How can we define $\hat{f}(A_1, \ldots, A_n)$?

Given a t-norm T, the *n-ary T-extension of f*, denoted $\hat{f}_T$ is defined as

$$\mu_{\hat{f}_T(A_1,\ldots,A_n)}(y) = \sup\{T(\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)) \mid x_i \in X_i \text{ and } y = f(x_1, \ldots, x_n)\}.$$

Representation by Means of α-Cuts

The following holds for all $\alpha \in [0, 1[$:

$$\widehat{f}_{T_{\mathbf{M}}}(A_1, \ldots, A_n)^{>\alpha} = f(A_1^{>\alpha}, \ldots, A_n^{>\alpha})$$

Note that this *does not hold* for $T \neq T_{\mathbf{M}}$!

Fuzzy Arithmetics

Let A_1 and A_2 be two fuzzy sets on the real numbers $\mathbb{R}$.

- The T-extension of the addition $f(x_1, x_2) = x_1 + x_2$ is called *T-addition (T-sum)*. We use the special notation

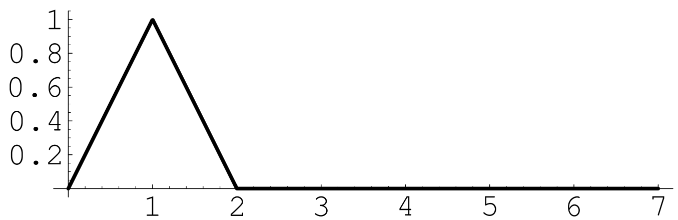
$$A_1 \oplus_T A_2 = \widehat{f}_T(A_1, A_2).$$

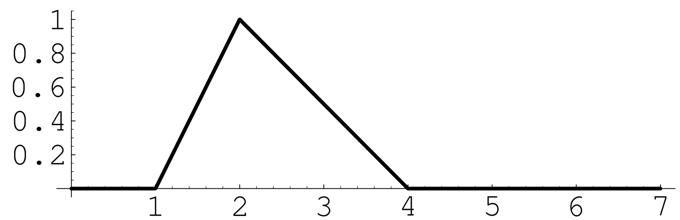
- The T-extension of the multiplication $f'(x_1, x_2) = x_1 \cdot x_2$ is called *T-multiplication (T-product)*. We use the special notation

$$A_1 \otimes_T A_2 = \widehat{f'}_T(A_1, A_2).$$

Example: Addition of Fuzzy Sets

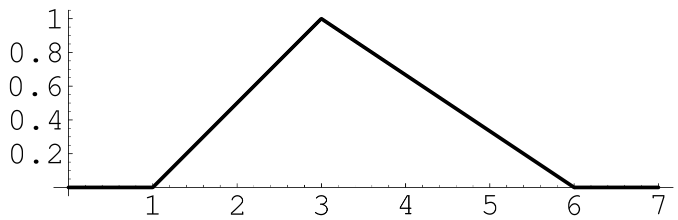
A_1





A_2

$A_1 \oplus_{T_{\mathbf{M}}} A_2$



Fuzzy Numbers

A fuzzy set A on the real numbers $\mathbb{R}$ is called a *fuzzy number* if it has the following properties:

Normality: there is an $x \in \mathbb{R}$ such that $\mu_A(x) = 1$

Convexity: every strict α-cut is an interval

Boundedness: all strict α-cuts are bounded

Some Remarks

- If x_0 is a value where $\mu_A(x_0) = 1$ for a fuzzy number A, then the membership function μ_A is non-decreasing to the left of x_0 and non-increasing to the right of x_0
- A fuzzy set A on $\mathbb{R}$ is convex if and only if the following holds for all chains $x \leq y \leq z$:

$$\mu_A(y) \geq \min(\mu_A(x), \mu_A(z))$$

- Some authors call our notion of a fuzzy number a *fuzzy interval* and use the term “fuzzy number” under additional, more restrictive assumptions

Trapezoidal Fuzzy Numbers

A chain of real numbers $a < b \leq c < d$ defines a *trapezoidal fuzzy number* A in the following way:

$$\mu_A(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x < b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c < x \leq d \\ 0 & \text{if } d < x \end{cases}$$

Triangular Fuzzy Numbers

A chain of real numbers $a < b < c$ defines a *triangular fuzzy number* A in the following way:

$$\mu_A(x) = \max\Big(\min\big(\frac{x-a}{b-a}, \frac{c-x}{c-b}\big), 0\Big)$$

Note that triangular fuzzy numbers are nothing else but trapezoidal fuzzy numbers for which $b = c$ holds.

Adding Trapezoidal Fuzzy Numbers

Assume that we have two trapezoidal fuzzy numbers A_1 and A_2 derived from two chains $a_1 < b_1 \leq c_1 < d_1$ and $a_2 < b_2 \leq c_2 < d_2$, respectively.
Then the $T_{\mathbf{M}}$-sum of A_1 and A_2 is a trapezoidal fuzzy number derived from the chain

$$a_1 + a_2 < b_1 + b_2 \leq c_1 + c_2 < d_1 + d_2.$$

Adding Triangular Fuzzy Numbers

Assume that we have two triangular fuzzy numbers A_1 and A_2 derived from two chains $a_1 < b_1 < c_1$ and $a_2 < b_2 < c_2$, respectively.

Then the $T_{\mathbf{M}}$-sum of A_1 and A_2 is a triangular fuzzy number derived from the chain

$$a_1 + a_2 < b_1 + b_2 < c_1 + c_2.$$

Final Remarks

- The last two assertions only hold for the $T_{\mathbf{M}}$-extension, not for any other t-norms.
- The Yager family is the only class of t-norms that preserves the trapezoidal/triangular shape. In case $T \neq T_{\mathbf{M}}$, however, the formulas are more complicated.
- Other t-norms do not even preserve the piecewise linear shape.
- Multiplication of fuzzy numbers does not preserve the piecewise linear shape, no matter which t-norm we choose for the extension.