# Unit 2

## Fuzzy Sets and Fuzzy Logical Operations

# What Do We Know So Far?

We know that fuzzy logic is a generalized kind of logic which uses the unit interval $[0, 1]$ as the set of truth values.

We know that a fuzzy set on a universe $X$ is represented by a membership function which maps each element $x \in X$ to a degree of membership from the unit interval $[0, 1]$—directly generalizing characteristic functions.

# What Do We Need?

In order to be able to proceed IF-THEN rules involving vague linguistic expressions which are modeled by fuzzy sets, we need to have proper generalizations of logical operations and an inference scheme.

Let us start with the first question: *How can we extend the classical logical operations* $\wedge, \vee, \neg$ *to the unit interval* $[0, 1]$*?*

# Repetition: Basic Properties of Logical Operations

1. $p \wedge q = q \wedge p$, $p \vee q = q \vee p$ (commutativity)
2. $p \wedge (q \wedge r) = (p \wedge q) \wedge r$, $p \vee (q \vee r) = (p \vee q) \vee r$ (associativity)
3. $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ (distributivity)
4. $p \wedge 1 = p$, $p \vee 0 = p$ (neutral elements)
5. $p \wedge 0 = 0$, $p \vee 1 = 1$ (absorption)
6. $p \wedge p = p$, $p \vee p = p$ (idempotence)
7. $\neg(\neg p) = p$ (involution)
8. $\neg(p \wedge q) = \neg p \vee \neg q$, $\neg(p \vee q) = \neg p \wedge \neg q$ (De Morgan laws)
9. $p \wedge \neg p = 0$, $p \vee \neg p = 1$ (excluded middle)

Questions: *Which of them do we need? Which of them can we pre-*

# Standard Requirements for Fuzzy Conjunctions

1. Commutativity
2. Associativity
3. Non-decreasingness
4. $1$ is neutral element

An operation fulfilling these requirements is called triangular norm (t-norm).

# Triangular Norms

A mapping $T : [0,1]^2 \longrightarrow [0,1]$ is a triangular norm (t-norm) if it has the following properties (for all $x, y, z \in [0,1]$):

| | |
|---|---|
| Commutativity: | $T(x,y) = T(y,x)$ |
| Associativity: | $T(x, T(y,z)) = T(T(x,y), z)$ |
| Non-decreasingness: | $x \leq y \;\Rightarrow\; T(x,z) \leq T(y,z)$ |
| Neutral element: | $T(x,1) = x$ |

# The Four Standard t-Norms

$$T_{\mathbf{M}}(x, y) = \min(x, y)$$

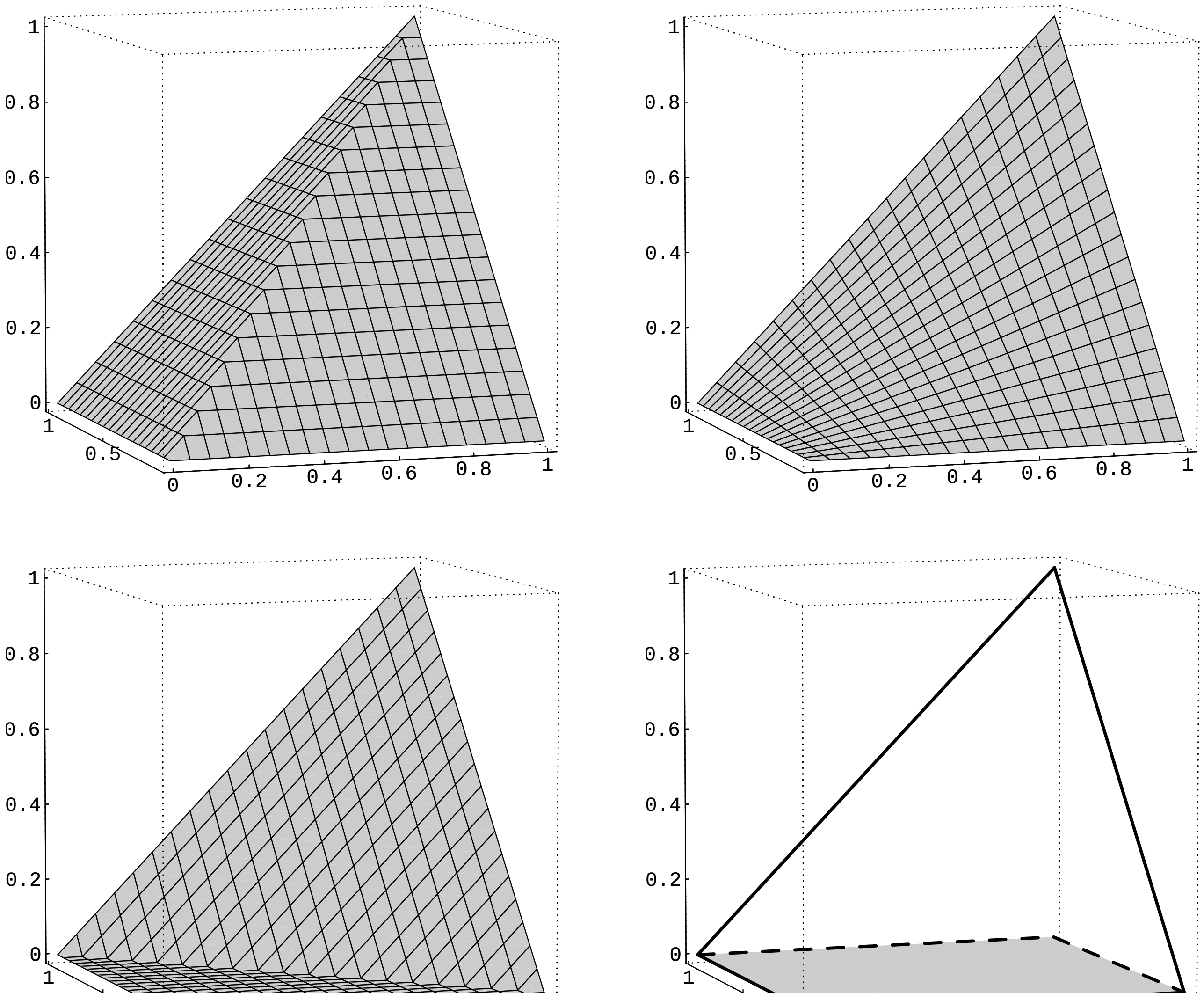
$$T_{\mathbf{P}}(x, y) = x \cdot y$$

$$T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0)$$

$$T_{\mathbf{D}}(x, y) = \begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

# The Four Standard t-Norms (Cont’d)

$T_{\mathbf{M}}$ $T_{\mathbf{P}}$




$T_{\mathbf{L}}$ $T_{\mathbf{D}}$

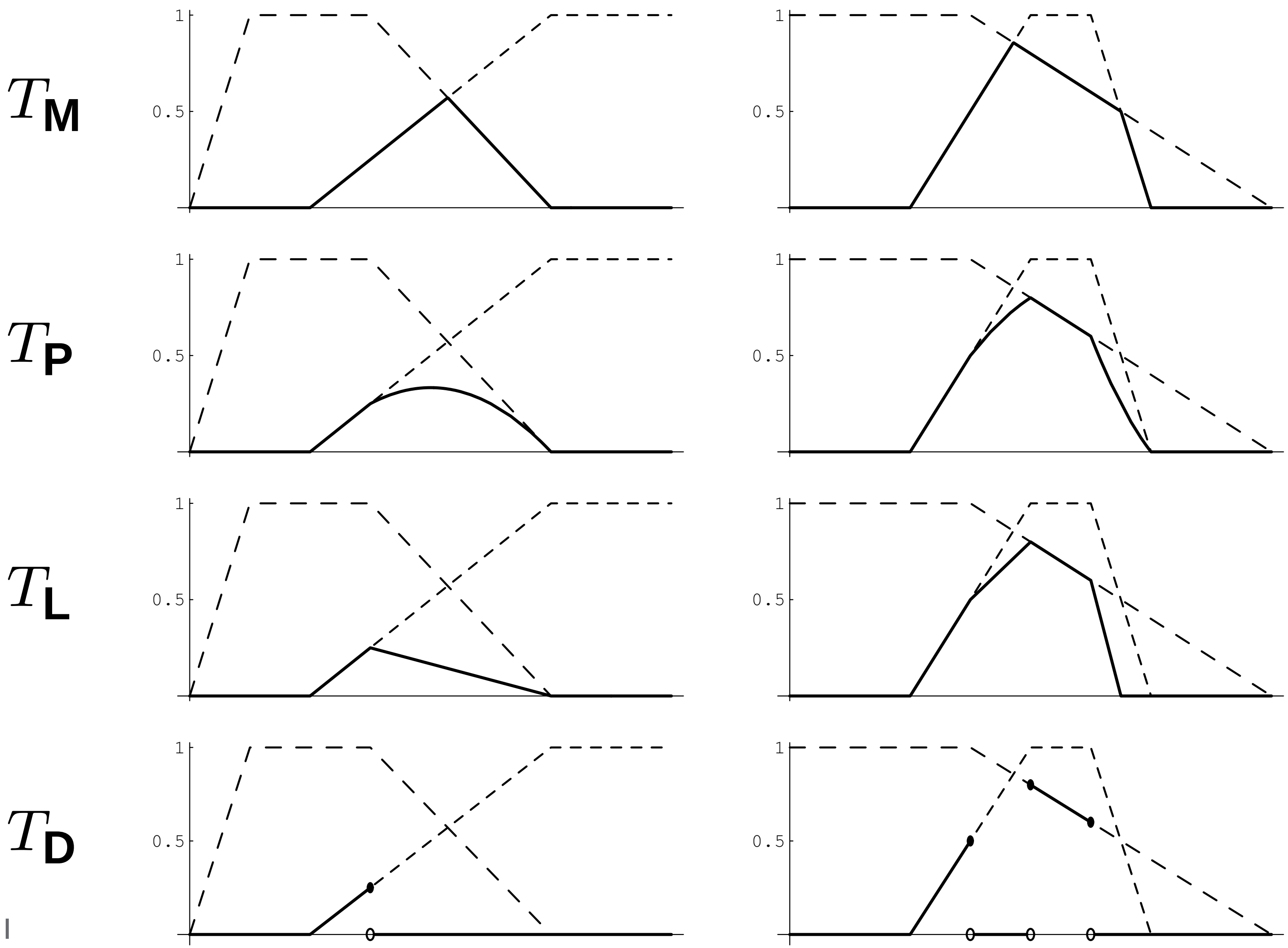
# Some Observations

- For all $x, y \in [0, 1]$, we have:

$$T_{\mathbf{D}}(x, y) \leq T_{\mathbf{L}}(x, y) \leq T_{\mathbf{P}}(x, y) \leq T_{\mathbf{M}}(x, y)$$

- It is easy to check that $T_{\mathbf{M}}$ is the largest possible t-norm and that $T_{\mathbf{D}}$ is the smallest possible t-norm
- $T_{\mathbf{M}}$ is the only t-norm fulfilling idempotence ($T(x, x) = x$)
- All except $T_{\mathbf{D}}$ are continuous
- $T_{\mathbf{P}}$ is the only differentiable one
- $T_{\mathbf{P}}$ is the only one that is strictly non-decreasing

# Examples of Intersections

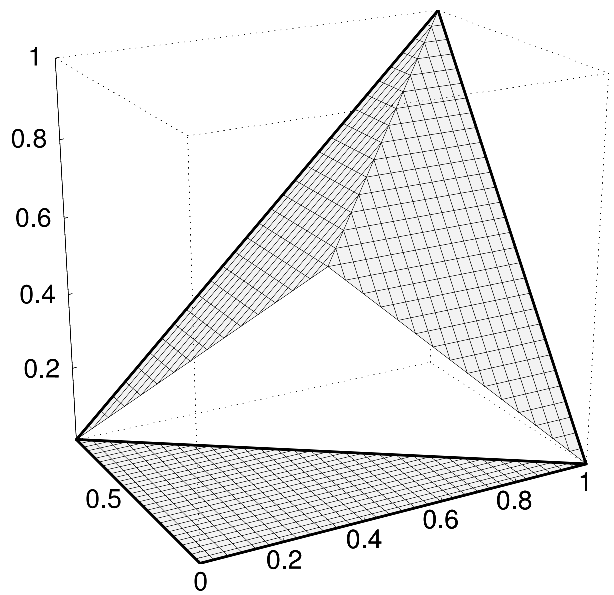


# Some Historical Background

- Triangular norms have emerged from the field of probabilistic metric spaces (Menger, Schweizer & Sklar)
- Have been studied much in the fields of associative functions and semigroups
- The introduction of t-norms into fuzzy set theory can be traced back to the late 1970ies (Antony/Sherwood, Alsina/Trillas/Valverde, Höhle/Klement/Weber)
- Since then, a rich theory has developed, also as a sub-field of fuzzy logic and fuzzy set theory

# Nilpotent Minimum

$$T_{\mathbf{nM}}(x,y) = \begin{cases} \min(x,y) & \text{if } x+y>1 \\ 0 & \text{otherwise} \end{cases}$$

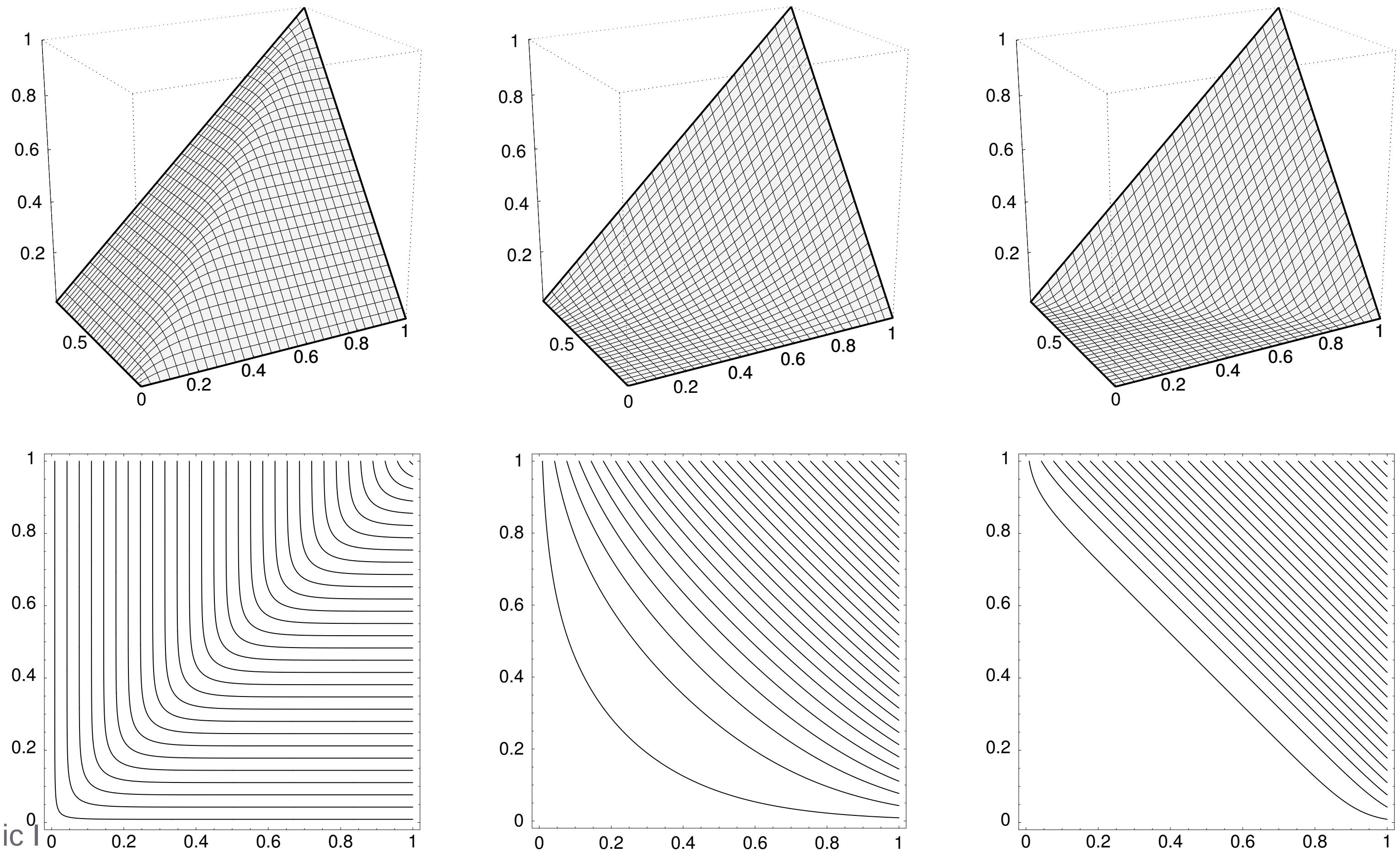


# The Frank Family $(T_\lambda^{\mathbf{F}})_{\lambda\in[0,\infty]}$

$$T_\lambda^{\mathbf{F}}(x,y) = \begin{cases} T_{\mathbf{M}}(x,y) & \text{if } \lambda = 0 \\ T_{\mathbf{P}}(x,y) & \text{if } \lambda = 1 \\ T_{\mathbf{L}}(x,y) & \text{if } \lambda = \infty \\ \log_\lambda \left(1 + \frac{(\lambda^x-1)(\lambda^y-1)}{\lambda-1}\right) & \text{if } \lambda \in \,]0,1[\,\cup\,]1,\infty[ \end{cases}$$

# The Frank Family (examples)

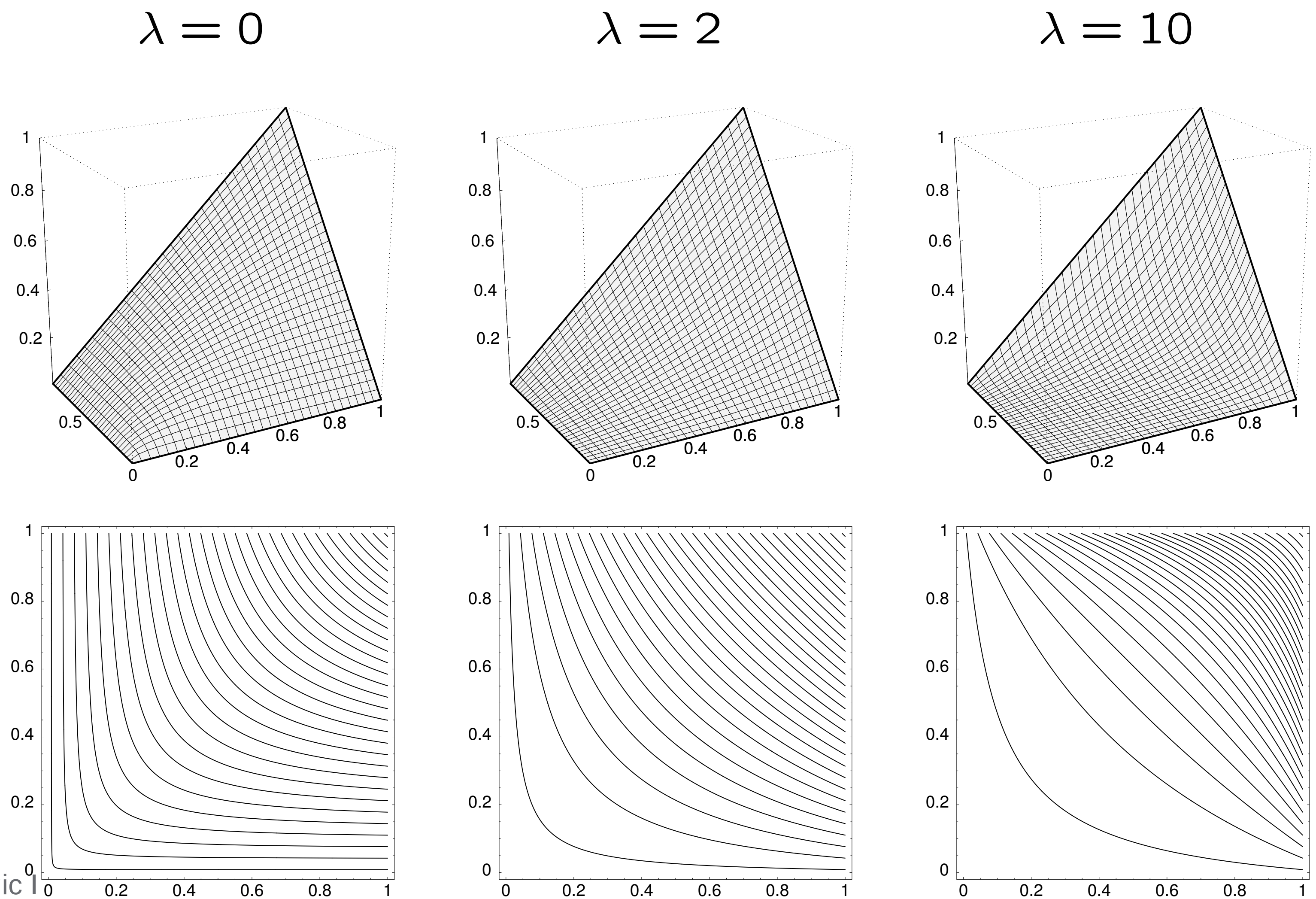
$\lambda = 10^{-9}$ | $\lambda = 100$ | $\lambda = 10^{9}$



# The Hamacher Family $(T_\lambda^{\mathbf{H}})_{\lambda\in[0,\infty]}$

$$T_\lambda^{\mathbf{H}}(x,y) = \begin{cases} T_{\mathbf{D}}(x,y) & \text{if } \lambda = \infty \\ 0 & \text{if } \lambda = x = y = 0 \\ \frac{xy}{\lambda+(1-\lambda)(x+y-xy)} & \text{if } \lambda \in [0,\infty[ \text{ and } (\lambda,x,y) \neq (0,0,0) \end{cases}$$

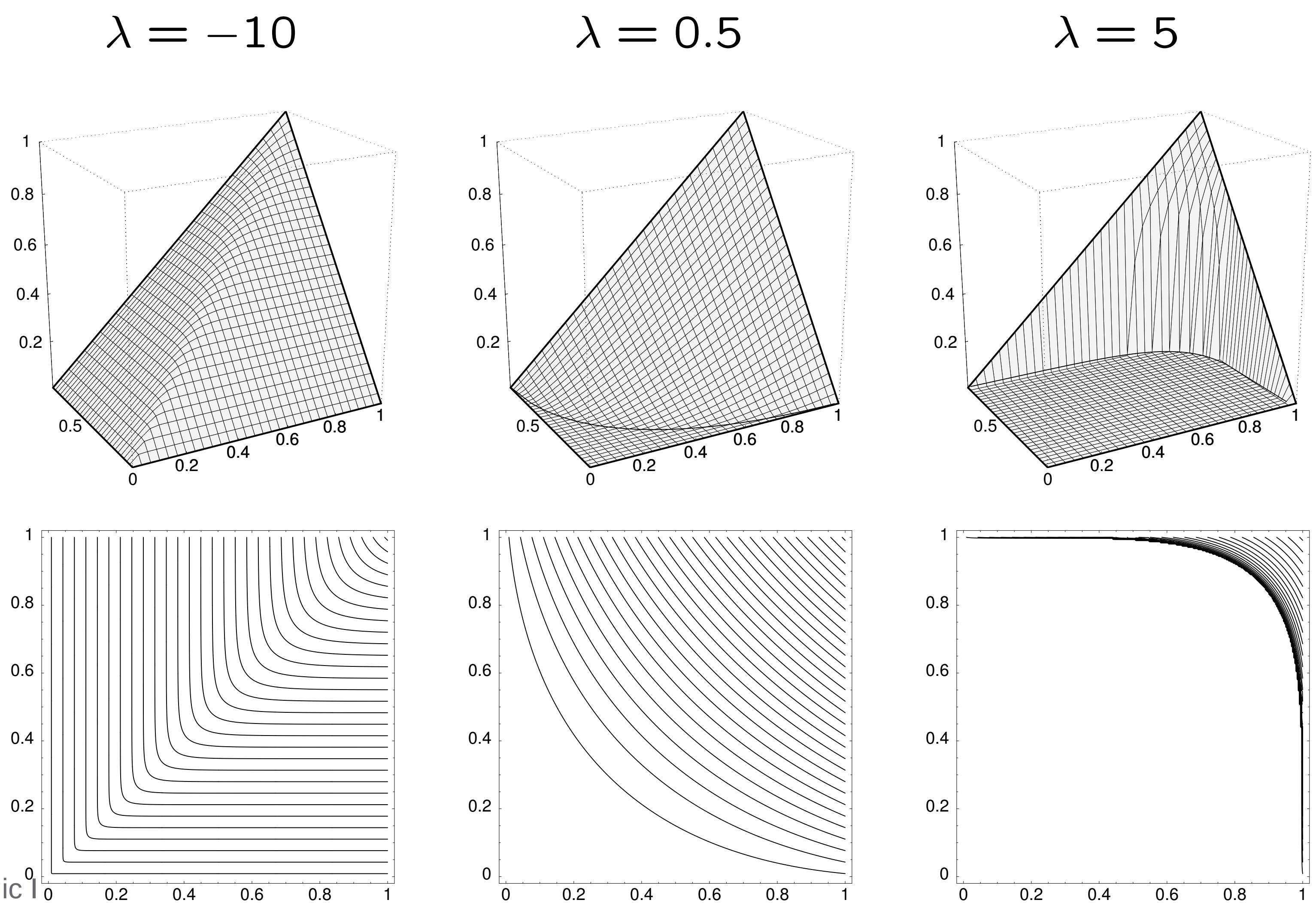
# The Hamacher Family (examples)



# The Schweizer-Sklar Family $(T_\lambda^{\mathbf{SS}})_{\lambda\in[-\infty,\infty]}$

$$T_\lambda^{\mathbf{SS}}(x,y) = \begin{cases} T_{\mathbf{M}}(x,y) & \text{if } \lambda = -\infty \\ T_{\mathbf{P}}(x,y) & \text{if } \lambda = 0 \\ T_{\mathbf{D}}(x,y) & \text{if } \lambda = \infty \\ \left(\max\left((x^\lambda + y^\lambda - 1), 0\right)\right)^{\frac{1}{\lambda}} & \text{if } \lambda \in \,]-\infty, 0[\, \cup \,]0, \infty[ \end{cases}$$

# The Schweizer-Sklar Family (examples)

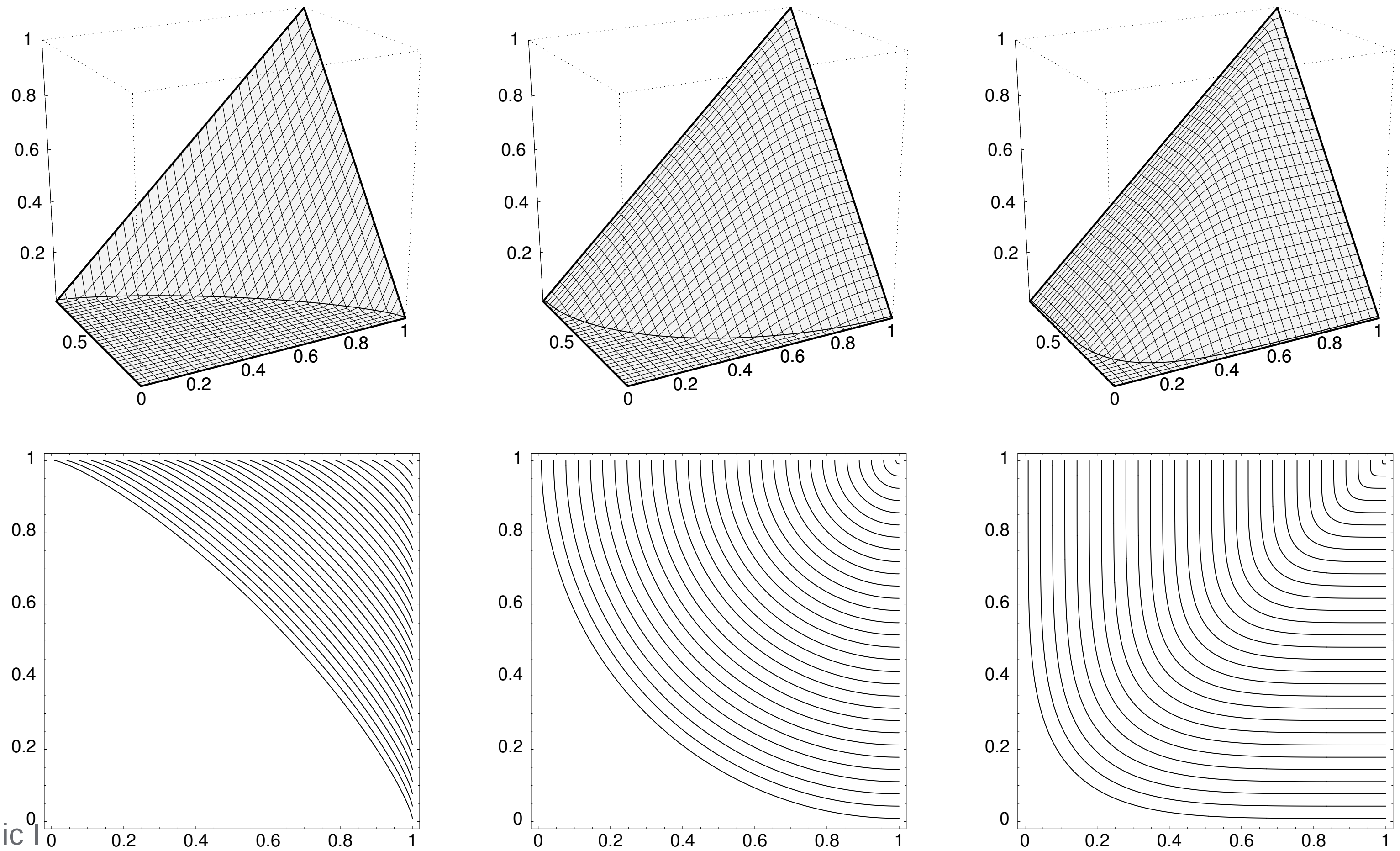


# The Yager Family $(T^{\mathbf{Y}}_{\lambda})_{\lambda\in[0,\infty]}$

$$T^{\mathbf{Y}}_{\lambda}(x,y) = \begin{cases} T_{\mathbf{D}}(x,y) & \text{if } \lambda = 0 \\ T_{\mathbf{M}}(x,y) & \text{if } \lambda = \infty \\ \max\left(1 - \left((1-x)^{\lambda} + (1-y)^{\lambda}\right)^{\frac{1}{\lambda}}, 0\right) & \text{if } \lambda \in \,]0,\infty[ \end{cases}$$

# The Yager Family (examples)

$\lambda = 0.8$ $\lambda = 2$ $\lambda = 5$



# How To Construct Triangular Norms?

If we take a strictly decreasing continuous mapping $f : [0,1] \to [0,\infty]$ which satisfies $f(1) = 0$, then

$$T_f(x, y) = f^{-1}\big(\min(f(0), f(x) + f(y))\big)$$

is a continuous triangular norm.

It can be shown (Schweizer & Sklar) that continuous Archimedean t-norms are uniquely characterized in this way, where a t-norm is called *Archimedean* if, for all $x, y \in ]0,1[$, there exists an $n \in \mathbb{N}$ such that

$$T(x, \ldots, x) < y.$$

# How To Construct Triangular Norms? (cont'd)

1. Continuous Archimedean t-norms the generator of which fulfills $f(0) = \infty$ are strictly non-decreasing on $]0, 1]^2$; such strictly non-decreasing continuous t-norms are called *strict*;
2. Continuous Archimedean t-norms the generator of which fulfills $f(0) < \infty$ have the property that for all $x \in ]0, 1[$ there exists an $n \in \mathbb{N}$ such that

   $$T(\underbrace{x, \ldots, x}_{n \text{ times}}) < 0;$$

   Continuous t-norms having this property are called *nilpotent*.

This means that continuous Archimedean t-norms are either strict or nilpotent, and that this choice is solely determined by the value of $f(0)$.

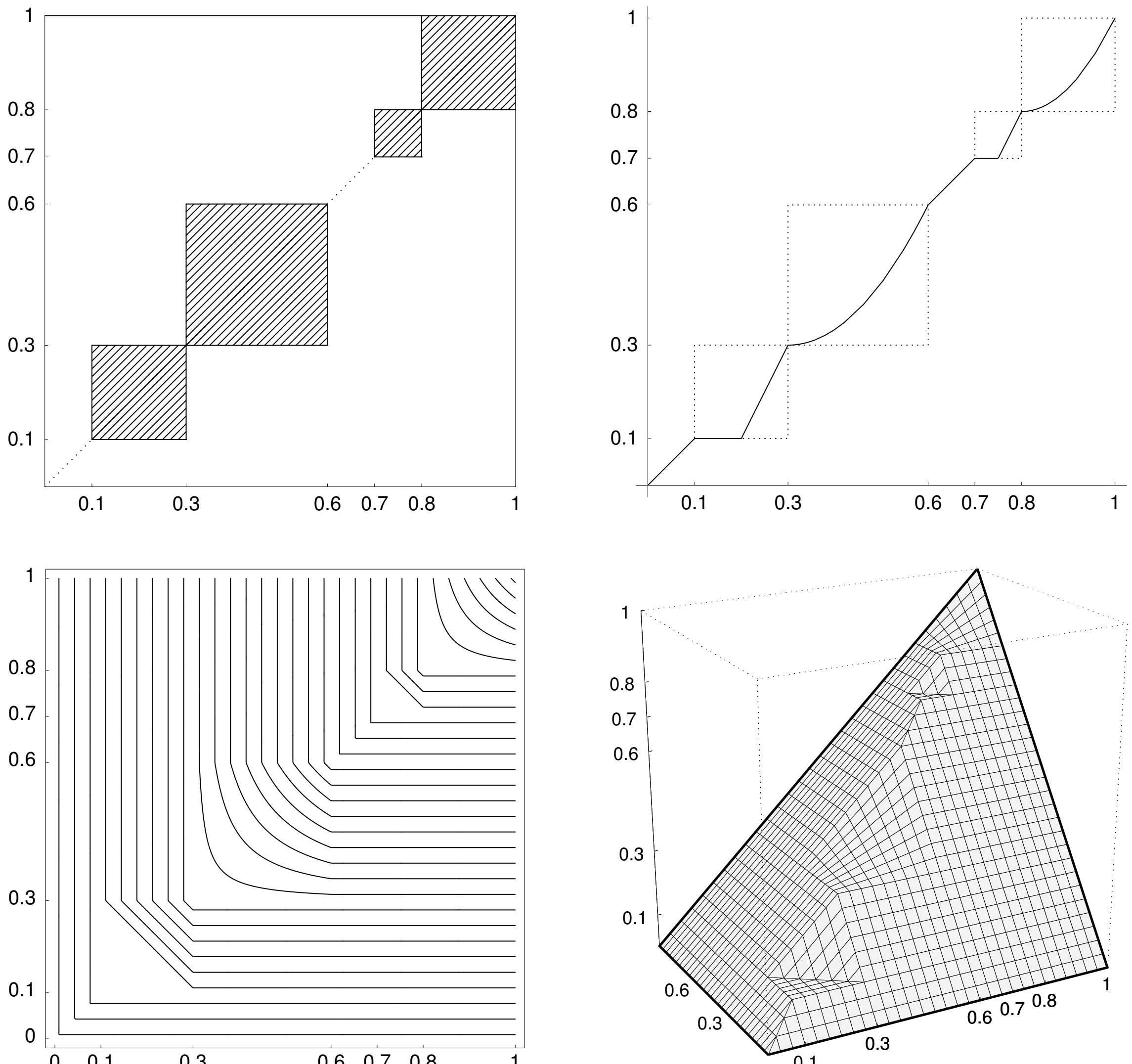
# Ordinal Sums

Let $(T_\alpha)_{\alpha\in A}$ be a family of t-norms and $(]a_\alpha, e_\alpha[)_{\alpha\in A}$ be a family of non-empty, pairwise disjoint open subintervals of [0, 1]. Then the following function $T : [0,1]^2 \longrightarrow [0,1]$ is a t-norm:

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha\left(\frac{x-a_\alpha}{e_\alpha-a_\alpha}, \frac{y-a_\alpha}{e_\alpha-a_\alpha}\right) & \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Note that all continuous t-norms are uniquely characterized as ordinal sums of continuous Archimedean t-norms.

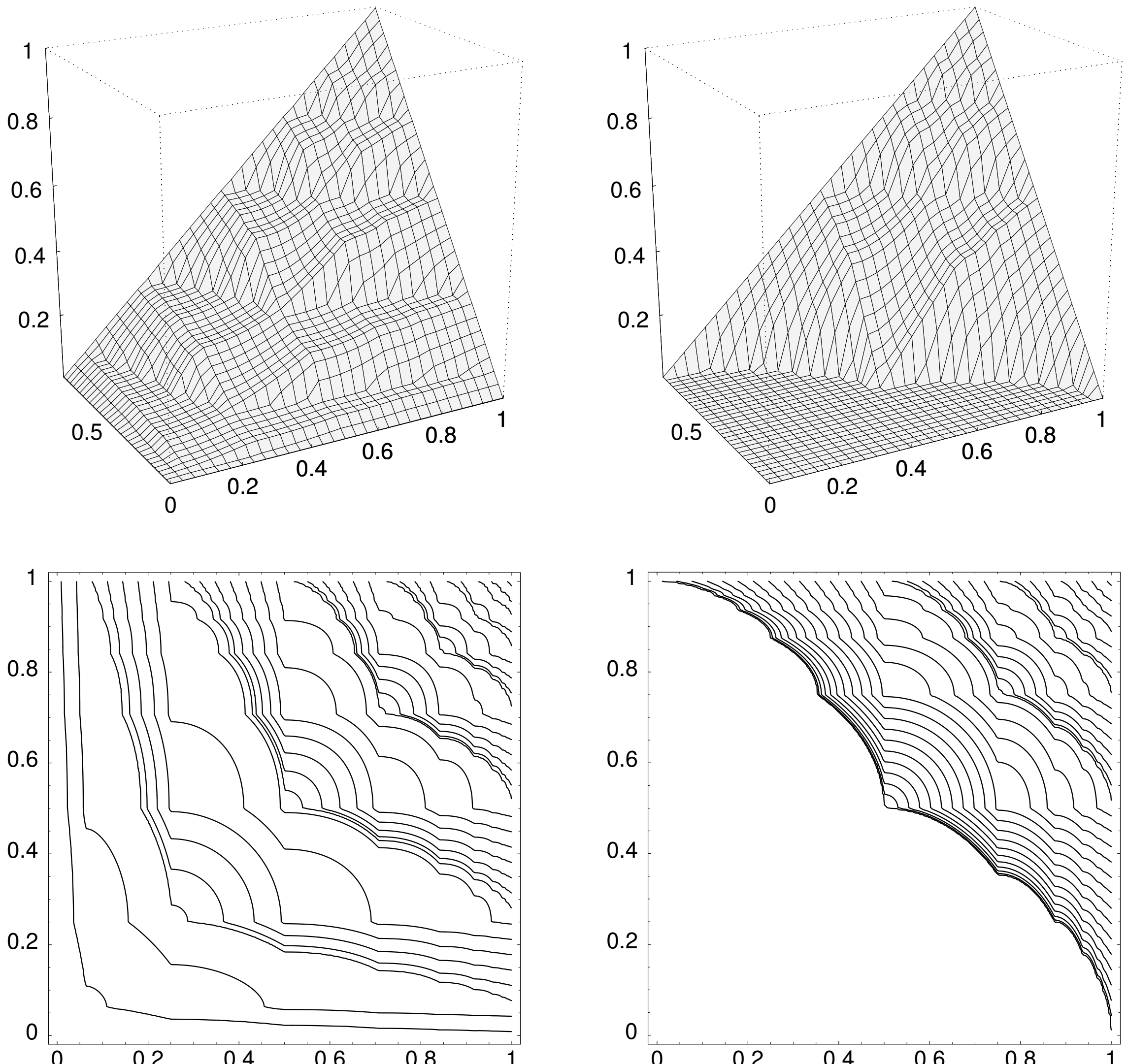
# Understanding Ordinal Sums Graphically



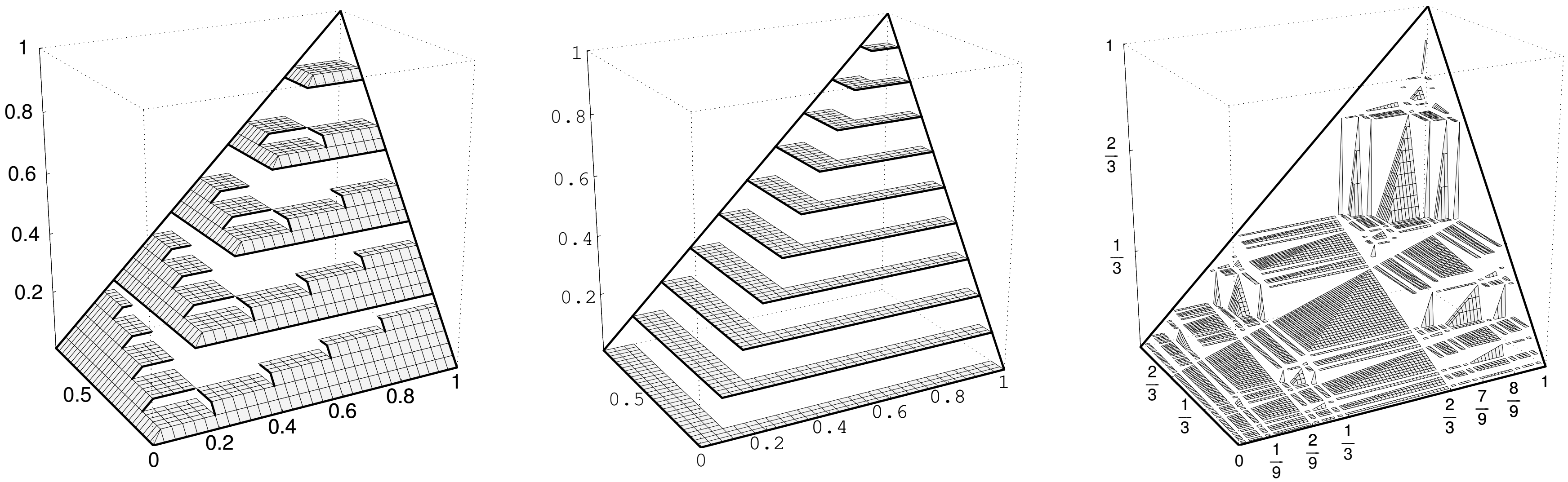
# Summary

- Triangular norms is a reasonable class of operations for modeling fuzzy conjunctions
- Triangular norms are delighting objects of study—much is known, but much is still unknown
- It is worth knowing about the mathematical background, however, in practical applications you will most often suffice with the three basic t-norms $T_{\mathbf{M}}$, $T_{\mathbf{P}}$, and $T_{\mathbf{L}}$

# “Strange Birds”



# “Strange Birds” (cont’d)



# Fuzzy Negations

A mapping $N : [0,1] \to [0,1]$ is called a *negation* if it is non-increasing and fulfills the boundary conditions $N(0) = 1$ and $N(1) = 0$.

A negation is called *strong* if it is continuous and strictly decreasing (therefore, bijective).

A strong negation is called *strict* if it is involutive, i.e.

$$N(N(x)) = x$$

for all $x \in [0,1]$. In other words, this means that $N^{-1} = N$.

# Fuzzy Negations: Examples

By far, the most common negation is the so-called *standard negation*

$$N_{\mathbf{S}}(x) = 1 - x.$$

It is easy to show that $N_{\mathbf{S}}$ is a strict negation.

Not very useful for practical applications, but of theoretical interest—the so-called *intuitionistic negation*:

$$N_{\mathbf{I}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

# Strict Fuzzy Negations: Representation

A negation $N$ is strict if and only if there exists a strictly increasing continuous mapping $\varphi : [0,1] \to [0,1]$ with $\varphi(0) = 0$ and $\varphi(1) = 1$ (a so-called $[0,1]$ *automorphism*) such that the following representation holds:

$$N(x) = \varphi^{-1}(1 - \varphi(x))$$

# Repetition: Basic Properties of Logical Operations

1. $p \wedge q = q \wedge p$, $p \vee q = q \vee p$ (commutativity)
2. $p \wedge (q \wedge r) = (p \wedge q) \wedge r$, $p \vee (q \vee r) = (p \vee q) \vee r$ (associativity)
3. $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ (distributivity)
4. $p \wedge 1 = p$, $p \vee 0 = p$ (neutral elements)
5. $p \wedge 0 = 0$, $p \vee 1 = 1$ (absorption)
6. $p \wedge p = p$, $p \vee p = p$ (idempotence)
7. $\neg(\neg p) = p$ (involution)
8. $\neg(p \wedge q) = \neg p \vee \neg q$, $\neg(p \vee q) = \neg p \wedge \neg q$ (De Morgan laws)
9. $p \wedge \neg p = 0$, $p \vee \neg p = 1$ (excluded middle)

Questions: *Which of them do we need? Which of them can we pre-*

# Standard Requirements for Fuzzy Disjunctions

1. Commutativity
2. Associativity
3. Non-decreasingness
4. $0$ is neutral element

An operation fulfilling these requirements is called triangular conorm (t-conorm).

# Triangular Conorms

A mapping $S : [0,1]^2 \longrightarrow [0,1]$ is a triangular conorm (t-conorm) if it has the following properties (for all $x, y, z \in [0,1]$):

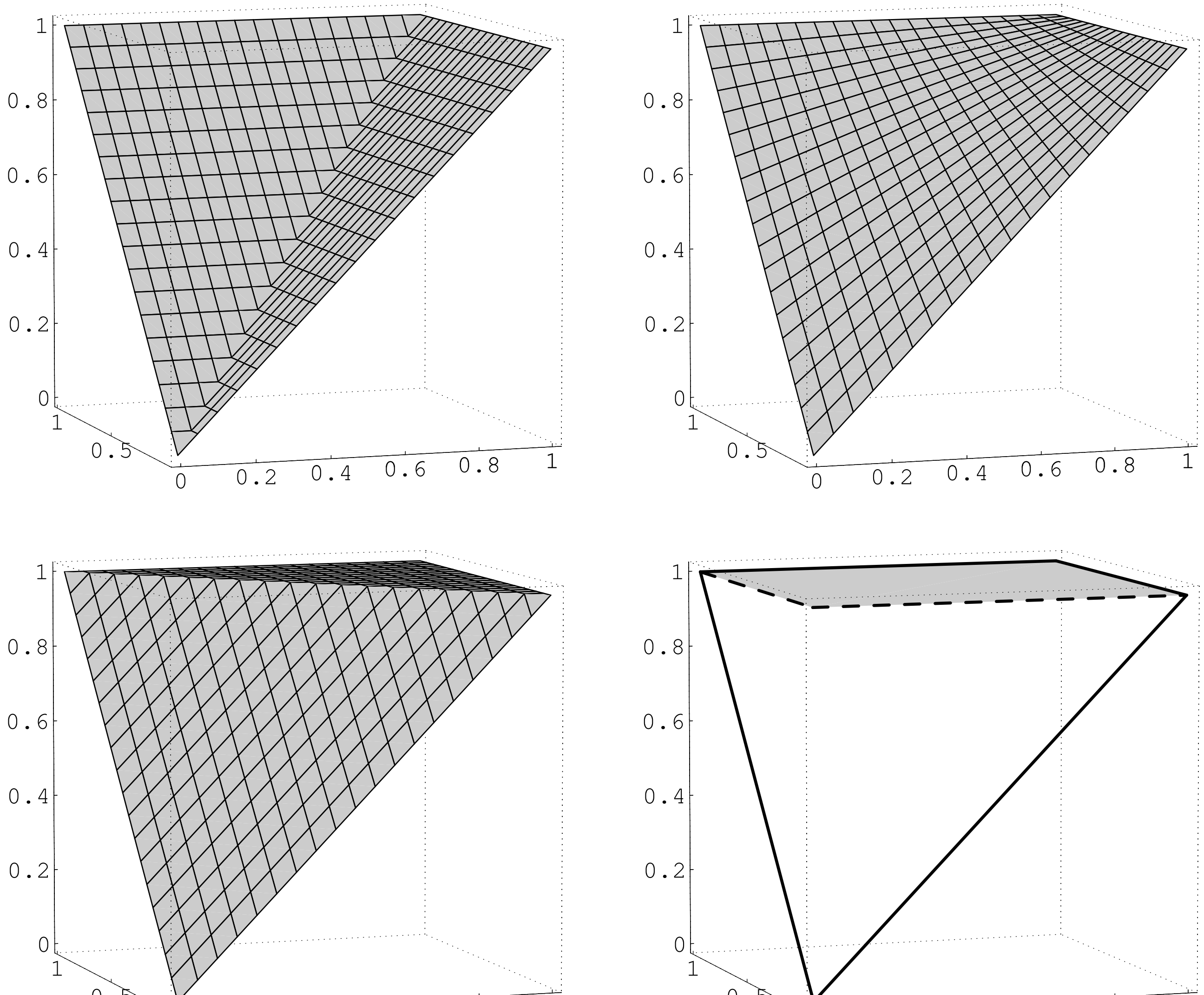
| | |
|---|---|
| Commutativity: | $S(x,y) = S(y,x)$ |
| Associativity: | $S(x, S(y,z)) = S(S(x,y),z)$ |
| Non-decreasingness: | $x \leq y \;\Rightarrow\; S(x,z) \leq S(y,z)$ |
| Neutral element: | $S(x,0) = x$ |

# The Four Standard t-Conorms

$$S_{\mathbf{M}}(x,y) = \max(x,y)$$
$$S_{\mathbf{P}}(x,y) = x + y - x \cdot y$$
$$S_{\mathbf{L}}(x,y) = \min(x+y,1)$$
$$S_{\mathbf{D}}(x,y) = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

# The Four Standard t-Conorms (Cont'd)

$S_{\mathbf{M}}$ $S_{\mathbf{P}}$




$S_{\mathbf{L}}$ $S_{\mathbf{D}}$

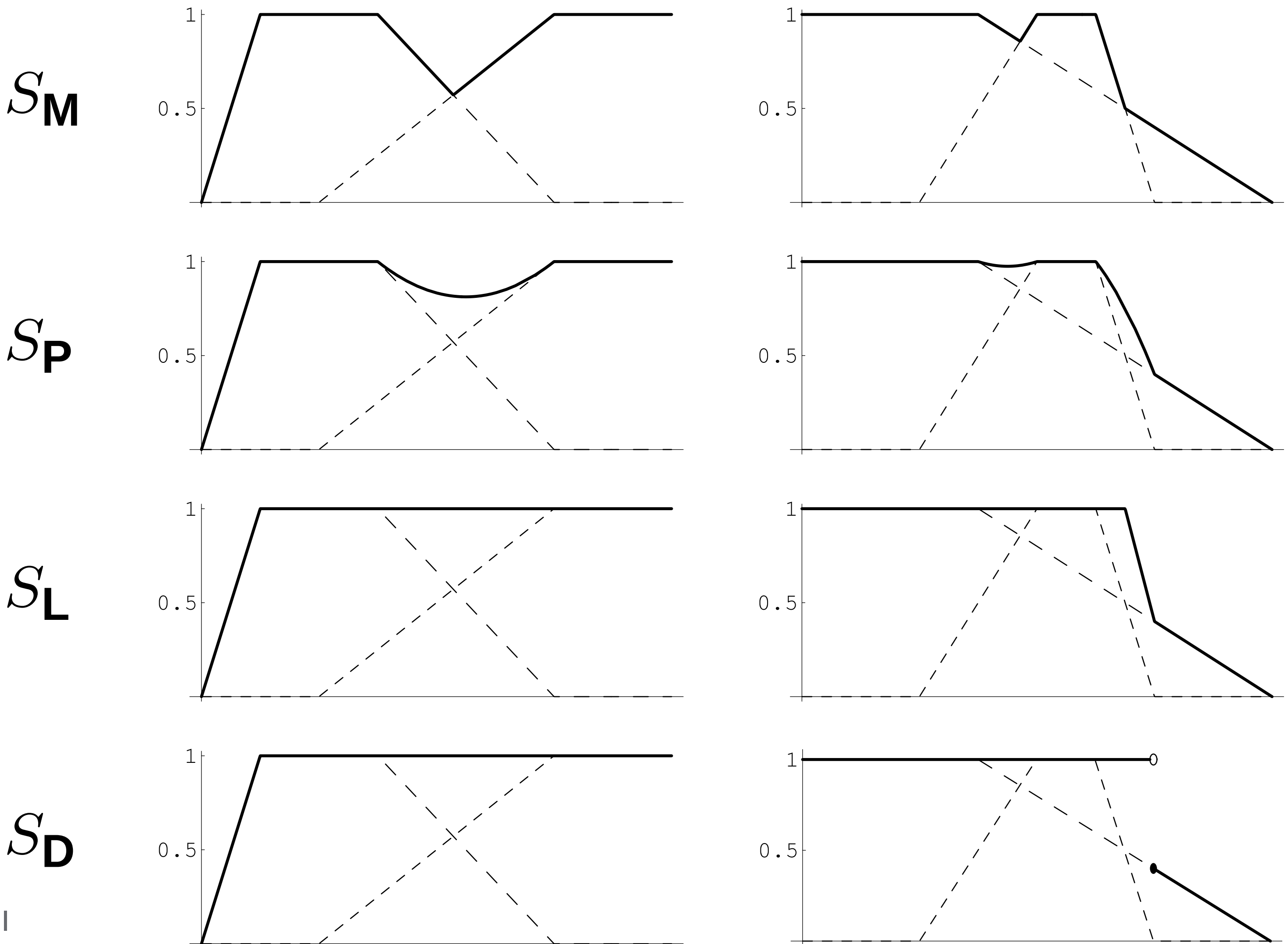
# Some Observations

- For all $x, y \in [0, 1]$, we have:

$$S_{\mathbf{M}}(x,y) \leq S_{\mathbf{P}}(x,y) \leq S_{\mathbf{L}}(x,y) \leq S_{\mathbf{D}}(x,y)$$

- It is easy to check that $S_{\mathbf{D}}$ is the largest possible t-conorm and that $S_{\mathbf{M}}$ is the smallest possible t-conorm
- $S_{\mathbf{M}}$ is the only t-conorm fulfilling idempotence ($S(x,x) = x$)
- All except $S_{\mathbf{D}}$ are continuous
- $S_{\mathbf{P}}$ is the only differentiable one
- $S_{\mathbf{P}}$ is the only one that is strictly non-decreasing

# Examples of Unions



# The Frank Family $(S^{\mathbf{F}}_{\lambda})_{\lambda\in[0,\infty]}$

$$S^{\mathbf{F}}_{\lambda}(x,y) = \begin{cases} S_{\mathbf{M}}(x,y) & \text{if } \lambda = 0 \\ S_{\mathbf{P}}(x,y) & \text{if } \lambda = 1 \\ S_{\mathbf{L}}(x,y) & \text{if } \lambda = \infty \\ 1 - \log_{\lambda}\left(1 + \frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1}\right) & \text{if } \lambda \in \,]0,1[\,\cup\,]1,\infty[ \end{cases}$$

# The Hamacher Family $(S_\lambda^{\mathbf{H}})_{\lambda \in [0,\infty]}$

$$S_\lambda^{\mathbf{H}}(x,y) = \begin{cases} S_{\mathbf{D}}(x,y) & \text{if } \lambda = \infty \\ 1 & \text{if } \lambda = 0 \text{ and } x = y = 1 \\ \frac{x+y+(\lambda-2)xy}{1+(\lambda-1)xy} & \text{if } \lambda \in \,]0,\infty[\, \text{ and } (\lambda,x,y) \neq (0,1,1) \end{cases}$$

# The Schweizer-Sklar Family $(S^{\mathbf{SS}}_{\lambda})_{\lambda\in[-\infty,\infty]}$

$$S^{\mathbf{SS}}_{\lambda}(x,y) = \begin{cases} S_{\mathbf{M}}(x,y) & \text{if } \lambda = -\infty \\ S_{\mathbf{P}}(x,y) & \text{if } \lambda = 0 \\ S_{\mathbf{D}}(x,y) & \text{if } \lambda = \infty \\ 1 - \left(\max\left(((1-x)^{\lambda} + (1-y)^{\lambda} - 1), 0\right)\right)^{\frac{1}{\lambda}} & \\ & \text{if } \lambda \in\, ]-\infty, 0[\, \cup\, ]0, \infty[ \end{cases}$$

# The Yager Family $(S_\lambda^{\mathbf{Y}})_{\lambda\in[0,\infty]}$

$$S_\lambda^{\mathbf{Y}}(x,y) = \begin{cases} S_{\mathbf{D}}(x,y) & \text{if } \lambda = 0 \\ S_{\mathbf{M}}(x,y) & \text{if } \lambda = \infty \\ \min\left(\left(x^\lambda + y^\lambda\right)^{\frac{1}{\lambda}}, 1\right) & \text{if } \lambda \in \,]0,\infty[ \end{cases}$$

# From t-Norms to t-Conorms and Back

Let $T$ be a t-norm and $N$ be a strong negation. Then

$$S(x, y) = N^{-1}\big(T(N(x), N(y))\big)$$

is a t-conorm. Observe: this is nothing else but the second De Morgan law:

$$N(S(x, y)) = T(N(x), N(y))$$

Correspondingly, if $S$ is a t-conorm, then

$$T(x, y) = N^{-1}\big(S(N(x), N(y))\big)$$

is a t-norm, which corresponds to the first De Morgan law:

$$N(T(x, y)) = S(N(x), N(y))$$

# De Morgan Triples and Dual Operations

In case that $N$ is a strict negation, the two De Morgan laws are equivalent.

A triple $(T, S, N)$ consisting of a t-norm $T$, a t-conorm $S$, and a strict negation $N$ such that the two De Morgan laws are fulfilled, is called a *De Morgan triple*.

We call a t-conorm $S$ and a t-norm $T$ *dual* to each other if they fulfill the De Morgan laws for the standard negation $N_{\mathbf{S}}$:

$$S(x, y) = 1 - T(1 - x, 1 - y)$$

$$T(x, y) = 1 - S(1 - x, 1 - y)$$

# Which Basic Properties of Logical Operations Hold Now?

1. $p \wedge q = q \wedge p$, $p \vee q = q \vee p$ (commutativity)
2. $p \wedge (q \wedge r) = (p \wedge q) \wedge r$, $p \vee (q \vee r) = (p \vee q) \vee r$ (associativity)
3. $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ (distributivity)
4. $p \wedge 1 = p$, $p \vee 0 = p$ (neutral elements)
5. $p \wedge 0 = 0$, $p \vee 1 = 1$ (absorption)
6. $p \wedge p = p$, $p \vee p = p$ (idempotence)
7. $\neg(\neg p) = p$ (involution)
8. $\neg(p \wedge q) = \neg p \vee \neg q$, $\neg(p \vee q) = \neg p \wedge \neg q$ (De Morgan laws)
9. $p \wedge \neg p = 0$, $p \vee \neg p = 1$ (excluded middle)

# Operations on Fuzzy Sets

Let $X$ be a non-empty universe and $(T, S, N)$ be a De Morgan triple. Then we can define the following three operations (with $A, B$ fuzzy sets on $X$):

$T$-intersection $A \cap_T B$:

$$\mu_{A \cap_T B}(x) = T(\mu_A(x), \mu_B(x))$$

$S$-union $A \cup_S B$:

$$\mu_{A \cup_S B}(x) = S(\mu_A(x), \mu_B(x))$$

$N$-complement $\complement_N A$:

# Implications

**$S$-implication:**

For a t-conorm $S$ and a negation $N$, we define

$$I_{S,N}(x,y) = S(N(x),y)$$

**Residual implication ($R$-implication):**

For a (left-)continuous t-norm $T$, we define

$$\vec{T}(x,y) = \sup\{u \in [0,1] \mid T(x,u) \leq y\}$$

# Examples

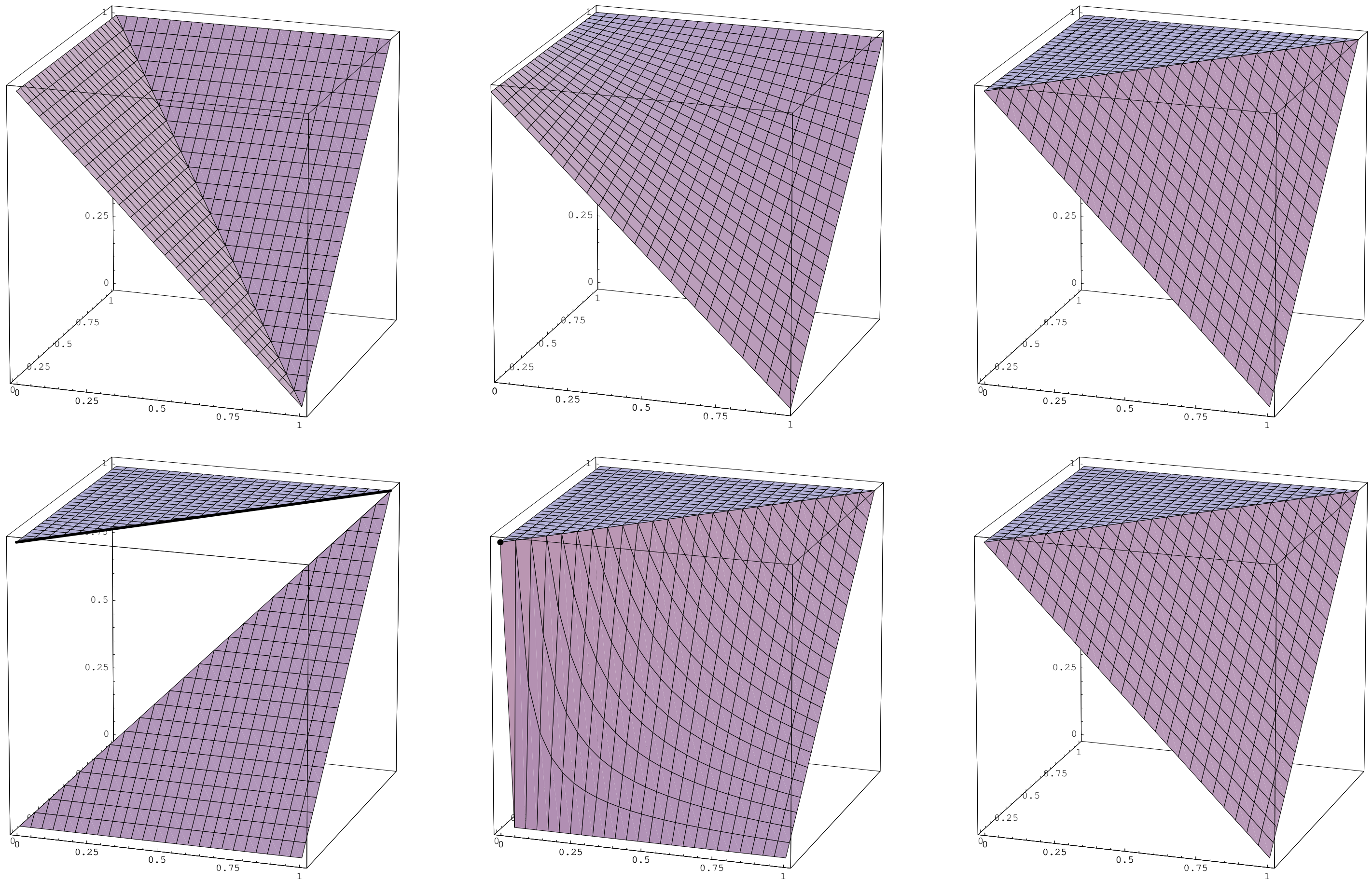
$$I_{S_{\mathbf{M}},N_{\mathbf{S}}}(x,y) = \max(1-x,y) \qquad \vec{T}_{\mathbf{M}}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

$$I_{S_{\mathbf{P}},N_{\mathbf{S}}}(x,y) = 1-x+x\cdot y \qquad \vec{T}_{\mathbf{P}}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$$

$$I_{S_{\mathbf{L}},N_{\mathbf{S}}}(x,y) = \min(1-x+y,1) \qquad \vec{T}_{\mathbf{L}}(x,y) = \min(1-x+y,1)$$

# Examples (cont'd)

$I_{T_{\mathbf{M}},N_{\mathbf{S}}}$ $\qquad$ $I_{T_{\mathbf{P}},N_{\mathbf{S}}}$ $\qquad$ $I_{T_{\mathbf{L}},N_{\mathbf{S}}}$



# Aggregation Operators

A function

$$\mathbf{A} : \bigcup_{n \in \mathbb{N}} [0,1]^n \longrightarrow [0,1]$$

is called an *aggregation operator* if it has the following properties:

1. $\mathbf{A}(x_1, \ldots, x_n) \leq \mathbf{A}(y_1, \ldots, y_n)$ whenever $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$
2. $\mathbf{A}(x) = x$ for all $x \in [0,1]$
3. $\mathbf{A}(0, \ldots, 0) = 0$ and $\mathbf{A}(1, \ldots, 1) = 1$

# Aggregation Operators: Examples

- All t-norms and t-conorms are aggregation operators (with the conventions $T(x) = x$ and $S(x) = x$)
- All weighted arithmetic and geometric means are aggregation operators
- Another prominent example—*ordered weighted average (OWA) operators*: Take a sequence $(x_1, \ldots, x_n)$ and sort it into a descending list $(\tilde{x}_1, \ldots, \tilde{x}_n)$. Then the corresponding OWA operator for a weight vector $\vec{w} = (w_1, \ldots, w_n)$ is defined as

$$\mathbf{OWA}_{\vec{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i \cdot \tilde{x}_i.$$